Yang Mills model of interacting particles in the classical field theory

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Abstract

The purpose is to study systems of interacting particles, in the General Relativity context, by the principle of least action, using purely classical concepts. The particles are described by a state tensor, accounting for the kinematic part (rotation) using a Clifford algebra, and the interaction part (charges). The force fields, including gravitation and other forces, are described by connections on principal bundles. A solution has been found to account in the lagrangian for individual, pointlike, particles. The constraints induced by the equivariance (gauge) and covariance are reviewed. Modified Lagrange equations are written for general lagrangians. A model, based on scalar products, Dirac operator and chirality is studied in more details. Problems related to symmetries, including the Higgs mechanism, are introduced.

With a comprehensive coverage of topics, and the purpose of finding a physical meaning to the mathematical tools used, it can be a useful pedagogical study. But it opens also some new paths.

CONTENTS

The principle of least action has been the workhorse of theoretical physics for decades. Both its high versatility and prodigious efficiency largely make up for the weakness of its foundations. In its many implementations two different paths can be broadly discerned. The classical approach (in newtonian or relativistic geometry) encompasses mechanics and the theory of fields, and provides sound basis to statistical mechanics and thermodynamics. There are many ways to address the lagrangian specification (Morrison [19], Soper [24]), but the key is to proceed quickly to the phase space, endowed with a symplectic structure, where all the mathematical tools can be efficiently deployed (Hofer [9]). The various Einstein-Vlasov equations are an example of this approach (Choquet-Bruhat [3]). On the other hand quantum mechanics and the quantum theory of fields make also an intensive use of the principle of least action, as an hamiltonian or lagrangian is required as starting point. The two main differences are that the distinction between particles (matter fields) and "force fields" (bosonic fields) is blurred, and that the basic axioms of quantum mechanics (such as summed up by Weinberg [30]) and the Wigner theorem open the way to a more direct analysis of the equations. It is the only theory that gives us some predictions for the physical characteristics of the particles and how they change but, if there is no need to aknowledge its power, we are still left with one the biggest enigma of modern physics : "Where does the first quantization come from ?". As both the classical and quantum approaches lead, through Poisson brackets and the likes, to Banach algebras, one way to answer this question is to circumvent the principle of least action and go straight to C*-algebra. It is roughly what is attempted with the algebraic quantum field theory (Halvorson [8]). One issue is that in the simplest of physical system (1 spinless particle) the set of observables is not a C*-algebra...and anyway one is still far away from understanding the axioms of quantum physics.

So, whatever one's personal philosophical belief about "realism in physics", it seems useful to pursue further the classical approach with the principle of least action. All the more so that decades of hard labour and progresses in mathematics have brought to us new schemes, such as the Yang-Mills description of the fields, and powerful tools such as fiber bundle or Clifford algebra. It is the main objective of this paper : check if it is possible to build a useful and comprehensive model of particles interacting with fields, in a purely classical way, using the tools of to day (or at least yesterday), hoping to get some hint at the meaning of quantum physics. Much work

has already been done about the mathematical foundations of a "modern" classical theory of fields (Giachetta [5]) but I want to focus here on putting together the ingredients to get the full picture of a physical system.

To be useful and comprehensive the model should :

- adopt the geometry of General Relativity, without any exotic feature (no extra dimension)

- describe the kinematic (meaning here rotation and any other self centered geometric movements), the dynamic (meaning here displacement in space-time) and the physical characteristics (such as mass and charge) of the particles, prior to any quantization

- include gravitation and "other" forces, seen separately (no "GUT" in stock) and treated as gauge fields in a Yang-Mills formalism

- stay at the "laboratory level" (no cosmology).

As much as possible the different mathematical objects and hypothesis should be clearly defined and related to physical or experimental procedures that could be used to get numerical values of the variables.

Two difficulties arise :

- The "point particle" issue : the need to manage simultaneously force fields and matter fields raises some mathematical difficulties which should not be discarded lightly. I overcome them with an adaptated Green function in what seems to be a new solution. However the self-radiation reaction issue (Poisson [21], Quinn [22]), which is more about solving the equations, has been kept out of the scope of the paper.

- The metric issue : the formalism of fiber bundle does not fit well with the traditional treatment of gravitation based upon the metric. Furthermore, as we will see, the connexion should not be torsion free. So I stay firmly in the scheme of fiber bundle and connection, expressed in the tetrads formalism, and the metric tensor is seen as a by-product of the orthonormal basis.

The first part gives the description of the geometric model (a gaussian normal coordinates system), the kinematic model (a representation of a Clifford algebra), the physical characteristics of the particles (through the representation of a unidentifed compact group U) ,the associated tensor bundle of their "state", and the covariant derivative.

The second part starts with a description of the configuration bundle and gives a solution to the treatment of individually interacting particles. Then it addresses the lagrangian issues : gauge invariance and general covariance, and sets up the most general constraints on a lagrangian.

The procedures to solve the variational problem are reviewed in the third

part, from the general fiber bundle and the functional derivative formalisms, the equations are listed and the Noether currents for gravitation and the other fields are evidenced with their super-potentials. The definitions and properties of the energy momentum tensor are reviewed including a "super-conservation law".

In the fourth part the scheme is implemented in a simple, but quite general, model. It requires the definition of scalar products on the fiber bundles, of the Dirac operator, and the introduction of chirality, which enables to further specify the representation spaces in the vector bundles. A lagrangian is specified. In this simplified, but still comprehensive, model the Noether currents lead to moments (linear and angular momentum, charge and "magnetic momentum") which characterize the particles. The gravitational connection has an explicit and simple formulation from the structure coefficients and the moments, showing that the connection will not usually be torsion free.

Symmetries are reviewed in the fifth part : the CPT problem has a simple explanation. Spatial symmetries (spin particles) induce a strong dependance of the state of the particles upon the 3 parameters defining a spatial rotation, and its "paradoxical" properties are evidenced. Physical symmetries (involving the group U) are studied first as defining families of particles, then through the breakdown of symmetries in the Higgs mechanism.

In the final part the model is implemented to do the junction with general relativity and electromagnetism.

Overall the paper shows most of the fundamental concepts of field theory in a consistent and comprehensive scheme, and in a fully classical picture. Implementing all the tools which are now common in theoretical physics, it should be a useful pedagogical instrument. This paper requires a common knowledge of the principles of fiber bundles, connections and Lie groups. As often as possible the basic definitions are recalled. Some calculations are a bit cumbersome, but I feel better to follow a simple if lengthy path than to risk shortcuts which would require sophisticated mathematical concepts.

As such the model can be a good starting point to investigate further

Part I
FOUNDATIONS

Our aim is to build a model describing a system of N-particles interacting with gravitation and other forces fields. The first key word is "system". In physics it has great implications : it means that the particles can be identified, their trajectories and properties measured and followed up for some time, by observers who are networked in order to get a full picture of what is happening. Perhaps some people would object that it is an impossible task from the quantum point of view, but in classical physics, where we stand, it is a sensible one, and any scientist should start by defining what it means by the "system" that he (or she) is modelling. So the system is supposed to be included in a not too big region of the universe (that excludes cosmology), clearly defined in space and time (no infinities, but large enough so that it can be considered as isolated from external interactions), inhabited by particles (which, as usual, are physical objects without any internal strucures involved - no scale is implied), gravitation and "other" forces fields (electromagntic and the likes). A single, or a network, of observers has defined a map and frames to measure all geometric quantities (such as position, speed and angular momentum) and procedures to measure the physical quantities such as charge and fields. It is clear that the second are deduced from the first using some test particles or fields.

The first step is so to define the geometric part of the model.

1 GEOMETRY

The geometry is that of general relativity : the space-time universe is a smooth connected Hausdorff manifold M endowed with a metric g which has the signature - +++ or + - - -. The signature is not a trivial issue as we shall see. I will use the less conventional, but here more convenient (and more natural), - + + + signature.

In general relativity it is traditional to take g as a starting point. With the additional hypothesis that matter particles have a constant 4-velocity equal to -1 (with the - + + + signature) and photons a null 4-velocity one can build a causal structure of events over M which, with some generally accepted assumptions (called hyperbolicity), leads for M to the structure of

a trivial fiber bundle SxR where S is a 3-dimensional space-like hypersurface (a Cauchy surface) (Wald [29]).

Another way is to start from a bundle of orthonormal bases (the tetrads). The key ingredient is a principal fiber bundle O_M modelled on the connected component of the identity $SO_0(3,1)$ of the Lorentz group SO(3,1). Local trivializations charts are maps : $\varphi_o : M \times SO_0(3,1) \to O_M$. From it one builds an associated vector bundle $G_M = O_M \times_{SO_0(3,1)} R^4$ using the standard representation $(R^4, \jmath)$ of SO(3,1). The orthonormal bases $\partial_i(m)$ are the images of the canonical base ε_i of R^4 : $(\varphi_o(m,1), \varepsilon_i) \simeq \left(\varphi_o(m,h^{-1}), \jmath(h)_i^j \varepsilon_j\right)$ but with a Lorentz metric and are defined with respect to an holonomic frame by a matrix $[O] \in GL(4) : \partial_i(m) = O_i^\alpha \partial_\alpha$. The vector ∂_0 defines a time like distribution T(O) and the 3 vectors $\partial_1, \partial_2, \partial_3$ a space-like distribution S(O). Together they define a metric $g_{\alpha\beta} = \eta^{ij} O_i^\alpha O_j^\beta$ with which the basis is orthonormal. If this strucure has a physical meaning, the distributions are integrable over M and define a foliation of M by the hypersurfaces orthogonal to T, and we get back the previous topology with hyperbolicity. The necessary and sufficient condition for that is that the 1-form $O_0^\alpha dm^\alpha$ is closed. If M is simply connected then there is a scalar map $N(m) : O_0^\alpha dm^\alpha = (\partial_\alpha N)\, dm^\alpha$ [1]which gives a unique time to a point in the universe.

This second approach seems a bit abstract. But actually it is closer to the way an observer can see the structure of the universe. Indeed it is not sufficient to define mathematical objects : we should also give some procedure, however farfetched, to link them to physical measurements.

1.1 Building a chart

A point in the universe is situated by 4 components, and we need to build a map : $R^4 \to M$. It is easier as we have limited the system in a "not too big" region of M. This map is a classical "gaussian" chart. We recall how it works.

1) The starting point is a connected space-like hypersurface S(0) : it represents the "present" of an observer at the time t=0. The induced metric upon S(0) is riemanian. Over each point x of S(0) there is a unique unitary future oriented vector n(x) normal to S(0) and in a neighbourood of x there is a unique geodesic tangent to n(x) . So we can define a family of geodesics

[1] I will usually use the Einstein indices summation convention

$\gamma(x,t)$ going through x and tangent to n(x) and a vector field n(m) in each point m in the future of S(0). This vector field is the infinitesimal generator of diffeomorphisms from $x \in S(0)$ to $m = \exp tn(x)$ which for each $t \geq 0$ maps S(0) in a hypersurface S(t), the set of points in M for which the time coordinate is t. Let us prove that the vector field n(m) is orthonormal to S(t).

Let $\partial_i(x,0), i = 1,2,3$ be a set of othonormal bases in S(0) and an arbitrary holonomic basis $\partial_\alpha(m) \in T_m M$. The derivative $(\exp tn(x))'$ maps the basis $\partial_i(x,0)$ in a basis (exp is a diffeomorphism) $\partial_i(x,t) = O_i^\alpha \partial_\alpha$ of $T_{m(t)}S(t)$ in m(t)=exptn(x). Let $\phi(t) = g_{\alpha\beta}(m(t))n^\alpha(m(t))\,O_i^\beta(x,t)$ be the scalar product between n(m(t)) and $\partial_i(x,t)$ with the metric g(m(t)) and ∇ the covariant derivative on M.

We have :

$\frac{d}{dt}\phi(t) = n^\alpha(m(t))\nabla_\alpha\phi(t) = n^\alpha\nabla_\alpha\left(g_{\beta\gamma}n^\beta O_i^\gamma\right)$

$= n^\alpha g_{\beta\gamma}n^\beta\nabla_\alpha O_i^\gamma + n^\alpha g_{\beta\gamma}O_i^\gamma\nabla_\alpha n^\beta$

but :

$n^\alpha\nabla_\alpha n^\beta = 0$ because n(m) is tangent to a geodesic

$n^\alpha\nabla_\alpha O_i^\gamma = O_i^\alpha\nabla_\alpha n^\gamma$ because the vectors n et ∂_i are linearly independants

$\frac{d}{dt}\phi(t) = \partial_i^\alpha g_{\beta\gamma}n^\beta\nabla_\alpha n^\gamma$

$= O_i^\alpha\frac{1}{2}\left(g_{\beta\gamma}n^\beta\nabla_\alpha n^\gamma + g_{\gamma\beta}n^\gamma\nabla_\alpha n^\beta\right)$

$= O_i^\alpha\frac{1}{2}\nabla_\alpha\left(g_{\beta\gamma}n^\beta n^\gamma\right) = 0$ because $g_{\beta\gamma}n^\beta n^\gamma = -1$

So the scalar product $g_{\alpha\beta}(m(t))n^\alpha(m(t))\,O_i^\beta(x,t)$ is constant over m(t), it is null in t=0, and the vectors $\partial_i(x,t)$ are orthogonal with n and n is orthogonal with S(t).■

2) We can run in two troubles in the process. The geodesics may not be complete: they start or stop in some finite time. It is a singularity in the universe. Or the geodesics may cross each other. It is another singularity (which always exist : see Wald [29] chapt.9). But if we limit our system to a non-exotic region (without black-hole) and a sensible time period $[0,T]$ (no "big bang") we should not run into such troubles. So we can define our system as enclosed in a region $\Omega \subset M$ generated by exptn(x) from an open domain $\Omega(0)$ with compact closure in S(0). Ω is a connected 4-dimensional manifold, relatively compact and geodesically complete. With the trivialization map $\varphi_\Omega : \Omega(0) \times [0,T] \to \Omega :: m = \varphi_\Omega(y,t)$ it has also the structure of a trivial fiber bundle with base $\mathbb{R}^+$.

3) How could we experimentally build such a chart ? A good example is given by the GPS system (Ashby [1]). Spatial coordinates in S(0) can be measured by any conventional method, such as electromagnetic signals if one assume that light travels at constant speed. Notice that events occuring in S(0) cannot be reported "live" so a co-ordinated network of observers is required. A (not too big) "free body", which stays still without any other force than gravitation, travels on a geodesic. To be sure to stay on a geodesic, or more generally to know how much he deviates from a geodesic, an observer can follow the movements of a free body in his local frame : it can be done with accelerometers (as in the IPhone).

4) Notice that the "time" t is just a coordinate, wich is measured by coordinated clocks by each observer.

1.2 The system

The initial conditions are defined by their values over S(0). Let us check that the set of particles is well defined. They travel on their world-line m with a proper time τ specific to each of them, defined within an additive constant. We assume that Ω is large enough so that any particle entering the system will stay within (or disappear). Ω is a fiber bundle with base R so for each point $m(\tau)$ there is one unique time $t_M = \pi(m(\tau))$. The 4-velocity u of the particle is a time-like, future oriented vector (it is a matter particle) so it is projected over R by a positive scalar : $\frac{dt_M}{d\tau} = \pi'(m(\tau)) \frac{dq}{d\tau} = \pi'(m(\tau)) u > 0$ and the map $t_M(\tau)$ is one-to-one. If a particle is observed at some time t>0 it can be observed (short to disapear entirely) at any other time. This seems obvious but has a strong consequence in the relativistic context.

We will assume that the system is closed, in that there is no interaction from outside.

1.3 The principal bundle

Any physical quantity is eventually measured through changes in tensorial quantities in some vector bundle over M. So we need a procedure to define local frames at each point of M.

1) It comes naturally from the chart : in each point $x \in S(0)$ the local observer chooses an orthonormal (euclidean) frame $(\partial_i)_{i=1,2,3}$. The fourth

vector $\partial_0(x) = n(x)$ is parallel to the 4-velocity of the observer. The frame is parallel transported along the geodesic (so it stays orthonormal), to get at each $m \in \Omega$ a standard frame $\partial_i = O(m)_i^\alpha \partial_\alpha$ from which the local observer can deduce, in a knowledgeable way, its own local frame. The parallel transport being continuous the orientation is preserved : the bases have the same spatial and time orientation which is defined as direct. Experimentally the parallel transport is done by checking the movements of a test body (such as a gyroscop) in a local transported frame.

2) The mathematical objects are : the principal fiber bundle O_M ,base Ω, modelled over $\mathrm{SO}_0(3,1)$, with trivialization charts : $\varphi_o : \Omega \times SO_0(3,1) \to O_M$

the associated vector bundle G_M : $G_M = O_v \times_{SO_0(3,1)} R^4$ with the standard representation $(R^4, \jmath)$ of SO(3,1)

the local orthonormal basis $(\partial_i)_{i=0,..3} : \partial_i(m) = O_i^\alpha \partial_\alpha$

Notation 1 *:*

The greek letters will always refer to an holonomic basis $(\partial_\alpha)_{\alpha=0,..3}$, with an arbitrary chart, unless otherwise specified

The latin letters i,j,... will refer to the non holonomic orthonormal basis $\partial_i = O(m)_i^\alpha \partial_\alpha$. i=0,1,2,3

The latin letters a,b,..p,q will refer to bases of Lie algebras

Whenever necessary matrices are enclosed in backets : $[O] = [O]_i^\alpha$, $[O'] = [O]^{-1} = [O']_\alpha^i$

The dual holonomic basis is denoted : $dx^\alpha : dx^\alpha(\partial_\beta) = \delta_\beta^\alpha$

The dual non holonomic basis is denoted: $\partial^i : \partial^i(\partial_j) = \delta_j^i$

The fundamental form is : $\Theta = \sum_j \partial^j \otimes \partial_j$

The matrix (indexed over 0,1,2,3):

$$[\eta] = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The set of sections on a fiber bundle G_M will be denoted $\Lambda_0 G_M$, the set of vector fields over a manifold M is $\Lambda_0 TM$ and the set of r-forms over M is $\Lambda_r TM^*$.

1.4 Gauge equivariance

The principle of relativity says that the laws of physics do not depend on the observer : if two observers study the same system, using different sets of frames (or gauge), their measurements can be deduced using the mathematical relations transforming one set of frame into the other. This is the practical meaning of the gauge equivariance. Let us see what can be transformed in the model.

1) The gaussian chart $\varphi_\Omega : \Omega(0) \times [0,T] \to \Omega$,the trivializations φ_o : $\Omega \times SO_0(3,1) \to O_M$ and the frames $\partial_i(m)$ are experimental constructions. But from them any observer can choose other charts or frames, following procedures and rules which can be made known to any other observer, so that they can compare their data.

2) In the theory of fields, quantum or classical, it is generally assumed that the law of physics are local (if the entanglement of particles has questionned this point, all gauge theories are local). It follows that all physical quantities defined over M take the mathematical form of sections of a bundle associated with O_M. So all physical quantities which can be expressed as tensor must belong to some vector bundle, modelled over a vector space which is a representation of the group defining the geometry, here SO(3,1). This is a useful but very general presciption, as there are infinitely many representations of the same group and not all have a physical meaning.

3) The frames are parallel transported, so their transformations must be continuous and the orientation preserved. But one can consider a *global* tranformation, involving the other connected components of SO(3,1). They are defined with one of the 3 matrices :

$$T = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} ; P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} ; PT = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

The frame is transformed according to : $\widetilde{\partial}_i(m) = J_i^j \partial_i(m)$ with $[J] = T, P, PT$.

P is the "space-inversion" matrix : it reverses the orientation of the space frame (it can be done by using another orientation on S(0)). T is the "time-reversal" matrix : it changes t in -t in the equations (the time is measured from 0 to $-\infty$). Experiments show that they are not physically admissible gauge transformations. In both cases the orientation of the space-time is reversed : so it leads to the conclusion that the 4 dimensional physical universe is oriented.

The third matrix reverse space and time orientations, but preserves the space-time orientation. Experiments show that this is an admissible transformation if simultaneously particles are changed into antiparticles (the "C" symmetry). We will come back to these matters in the fifth part.

4) Another open choice is the signature of the metric : at first there is no physical justification for using - + + + or + - - -.The linear groups SO(1,3) and SO(3,1) are identical, but manifolds equipped with one or the other metric are not isometric, and the Clifford algebra Cl(3,1) and Cl(1,3) are not isomorphic.

5) The choice of the Cauchy hypersurface S(0) is crucial : another surface defines another system, with possibly different particles and fields.

6) The model can be easily transposed to Special Relativity and Galilean Geometry.

In Special Relativity the geodesics are straigth lines, S(t) are hyperplanes, observers are inertial with constant 4-velocity u. The choice of the system is the choice of an hyperplane plane S(0) and its unitary future oriented normal u, and it fully determines the physical content of the system. The gauge group is the restricted Poincaré group, the semi-product of the orthochronous Lorentz group (the component of the identity of SO(3,1)) and the group of translations in the Minkovski space. The time t is just a coordinate, without specific meaning.

In Galilean geometry the universe can be viewed as a 4-dimensional affine space. The hypersurfaces S(t) are parallel and there is only one 4-dimensional velocity. Time becomes a physical, independant variable, identical for all the observers, defined within an affine similitude. The gauges groups are on one hand the semi direct product of SO(3), the group of rotations in

3-dimensional space, and of the group of translations in the 3-dimensional affine space, and on the other hand the group of similitudes in time.

2 PARTICLES

A particle is a point-like physical object moving along a world line with a constant, future oriented, 4-dimensional velocity. Experience shows that, besides any specific assumption about their "internal" structure, particles come with "states" depending on:

- how they "rotate" around their center mass in the universe : we will call these movements the kinematic part of the state

- how they behave when interacting with force fields : mass, charge,...We will call these their "physical characteristics".

These characteristics are modelled independantly. As we are in the classical picture they are not quantized, so there is no need to distinguish different kinds of particles.

2.1 Kinematic

The most efficient way to model the kinematic part is by using a Clifford algebra (Lasenby [17]) which gives a natural foundation for all kinds of spinors.

2.1.1 Clifford Algebra

1) Let F be a real vector space endowed with a symetric bilinear non degenerated function $\langle\rangle$ valued in the field K. The Clifford algebra Cl(F,$\langle\rangle$) and the canonical map $\imath : F \to Cl(F, \langle\rangle)$ are defined by the following universal property : for any associative algebra A with unit 1 and linear map $f : F \to A$ such that :

$\forall v \in F : f(v) \times f(v) = \langle v, v\rangle \times 1$

$\Leftrightarrow f(v) \times f(w) + f(w) \times f(v) = 2\langle v, w\rangle) \times 1$

there exists a unique algebra morphism : $\varphi : Cl(F, \langle\rangle) \to A$ such that $f = \varphi \circ \imath$

It always exists a Clifford algebra, isomorphic, as algebra, to the exterior algebra ΛF. Its internal product, noted by a dot $\cdot$ is such that :

$\forall v, w \in F : v \cdot w + w \cdot v = 2\langle v, w\rangle$

The Clifford algebra includes the scalar K and the vectors F. As vector space its bases can be taken as ordered products of vectors of an orthonormal basis (with $\langle\rangle$) of F and it has the dimension $2^{\dim F}$.

2) Clifford algebras built over the same vector space F are isomorphic if and only if the bilinear functions have the same signature. So Cl(3,1), Cl(1,3) over $\mathbb{R}^4, Cl(4,\mathbb{C})$ over $\mathbb{C}^4$,are not isomorphic. $Cl(4,\mathbb{C})$ is the complexified algebra of both Cl(3,1) and Cl(1,3).

Let $(\varepsilon_i)_{i=0,..3}$ the canonical basis of $\mathbb{R}^4$ (and $\mathbb{C}^4$ with complex components). In this basis an element of the Clifford algebra is written :

$w = s+x_1\varepsilon_1+x_2\varepsilon_2+x_3\varepsilon_3+x_0\varepsilon_0+y_3\varepsilon_1\cdot\varepsilon_2+y_2\varepsilon_1\cdot\varepsilon_3+y_1\varepsilon_2\cdot\varepsilon_3+z_1\varepsilon_1\cdot\varepsilon_0+z_2\varepsilon_2\cdot\varepsilon_0$
$+z_3\varepsilon_3\cdot\varepsilon_0+t_0\varepsilon_1\cdot\varepsilon_2\cdot\varepsilon_3+t_3\varepsilon_1\cdot\varepsilon_2.\varepsilon_0+t_2\varepsilon_1\cdot\varepsilon_3\cdot\varepsilon_0+t_1\varepsilon_2\cdot\varepsilon_3\cdot\varepsilon_0+u\varepsilon_1\cdot\varepsilon_2\cdot\varepsilon_3\cdot\varepsilon_0$

$(s,x_j,y_j,t_j,u)\in K\,(=\mathbb{R},\mathbb{C})$

but the internal product follows the fundamental relations which differ according to the scalar product :

Cl(4,C) : $\varepsilon_i\cdot\varepsilon_j+\varepsilon_j\cdot\varepsilon_i=2\delta_{ij}$

Cl(3,1) : $\varepsilon_i\cdot\varepsilon_j+\varepsilon_j\cdot\varepsilon_i=2\eta_{ij}$

Cl(1,3) : $\varepsilon_i\cdot\varepsilon_j+\varepsilon_j\cdot\varepsilon_i=-2\eta_{ij}$

3) Let N be $\mathbb{R}^4$ endowed with the bilinear function of signature - + + + and $\widetilde{\Upsilon}: N\to CL(4,C)$ the linear map defined by :

$j=1,2,3:\widetilde{\Upsilon}(\varepsilon_j)=\varepsilon_j;\widetilde{\Upsilon}(\varepsilon_0)=i\varepsilon_0$

The vectors v,v' in N read :

$v=x_1\varepsilon_1+x_2\varepsilon_2+x_3\varepsilon_3+x_0\varepsilon_0, v'=x_1'\varepsilon_1+x_2'\varepsilon_2+x_3'\varepsilon_3+x_0'\varepsilon_0$

and :

$\widetilde{\Upsilon}(v)=x_1\varepsilon_1+x_2\varepsilon_2+x_3\varepsilon_3+ix_0\varepsilon_0$

$\widetilde{\Upsilon}(v')=x_1'\varepsilon_1+x_2'\varepsilon_2+x_3'\varepsilon_3+ix_0'\varepsilon_0$

It is easy to check that :

$\widetilde{\Upsilon}(v)\cdot\widetilde{\Upsilon}(v')+\widetilde{\Upsilon}(v')\cdot\widetilde{\Upsilon}(v)=2\left(x_1x_1'+x_2x_2'+x_3x_3'-x_0x_0'\right)=2\left\langle v,v'\right\rangle_N$

So by the universal property of Clifford algebras there is a unique morphism $\Upsilon: Cl(3,1)\to Cl(4,C)$ such that $\widetilde{\Upsilon}=\Upsilon\circ\imath$ where ı is the canonical map $\imath: N\to Cl(3,1)$. The function Υ is not on-to : the image $\Upsilon\left(Cl(3,1)\right)$ is a sub-algebra Cl_c(3,1) in Cl(4,C) with elements :

$w=s+x_1\varepsilon_1+x_2\varepsilon_2+x_3\varepsilon_3+ix_0\varepsilon_0+y_3\varepsilon_1.\varepsilon_2+y_2\varepsilon_1.\varepsilon_3+y_1\varepsilon_2.\varepsilon_3+iz_1\varepsilon_1.\varepsilon_0+iz_2\varepsilon_2.\varepsilon_0+iz_3\varepsilon_3.\varepsilon_0$

$+t_4\varepsilon_1.\varepsilon_2.\varepsilon_3+it_3\varepsilon_1.\varepsilon_2.\varepsilon_0+it_2\varepsilon_1.\varepsilon_3.\varepsilon_0+it_1\varepsilon_2.\varepsilon_3.\varepsilon_0+iu\varepsilon_1.\varepsilon_2.\varepsilon_3.\varepsilon_0$

where the components s,u,... are real.

We will identify $\imath(N) \subset Cl(3,1)$ with N and $\widetilde{\Upsilon} = \Upsilon \circ \imath$ with Υ.

There is a similar result with N', $\mathbb{R}^4$ endowed with the bilinear function of signature + - - - and the map $\widetilde{\Upsilon}'(\varepsilon_j) = i\varepsilon_j; \widetilde{\Upsilon}'(\varepsilon_0) = \varepsilon_0$. The subalgebra $\Upsilon'(Cl(1,3)) = Cl_c(1,3) \neq Cl_c(3,1)$ but the two algebra have common elements such as $i\varepsilon_3 \cdot \varepsilon_0$. One goes, inside Cl(4,C), from images of Cl(3,1) to images of Cl(1,3) by the rule : $\Upsilon'(\varepsilon_j) = -i\eta^{jj}\Upsilon(\varepsilon_j)$ with $\eta^{00} = -1, j > 0 : \eta^{jj} = 1$

2.1.2 Spin Group

1) The involution $v \to -v$ on the vector space is extended in an involution α in the Clifford algebra. For homogeneous elements : $\alpha(v_{1.} \cdot v_2 .. \cdot v_r) = (-1)^r (v_1 \cdot v_2 .. \cdot v_r)$. It follows that the set of the elements of the Clifford algebra which are the sum of homogeneous elements which are themselves product of an even number of vectors is a subalgebra Cl_0. The complement vector space denoted Cl_1 is not a subalgebra. There is another involution : the transposition which acts on homogeneous elements by : $(v_1 \cdot v_2 ... \cdot v_r)^t = (v_r \cdot v_{r-1} ... \cdot v_1)$.

2) In Clifford algebra any element which is the product of non null norm vectors has an inverse :

$$(w_1 \cdot ... \cdot w_k)^{-1} = \alpha\left((w_1 \cdot ... \cdot w_k)^t\right) / \prod_{r=1}^{k} \langle w_r, w_r \rangle$$

The Spin group of a Clifford algebra is the subset of Cl_0 of elements which are the sum of an even product of vectors with norm +1. We will denote Spin(3,1) the connected component of the identity of the spin group of Cl(3,1) :

$Spin(3,1) = \{ \sum w_1 \cdot ..... \cdot w_{2k} : w_l \in N : \langle w_l, w_l \rangle_N = 1\}$

It is isomorphic to SL(2,C). Its identity is denoted 1.

3) As a group Spin(3,1) has a linear representation $(Cl(3,1), \mathbf{Ad})$ through the adjoint operator (denoted here in bold case to make the difference with the function Ad which is introduced below) :

$\mathbf{Ad} : Spin(3,1) \to \hom(Cl(3,1); Cl(3,1)) :: \mathbf{Ad}(s)\, w = s \cdot w \cdot s^{-1}$

which has the properties that the image a vector is still a vector and that it preserves the scalar product :

$v, w \in \mathbb{R}^4 \subset Cl(3,1) : \mathbf{Ad}(s)\, v \in \mathbb{R}^4; \langle v, w \rangle = \langle \mathbf{Ad}(s)\, v, \mathbf{Ad}(s)\, w \rangle$

Spin(3,1) is the universal covering group of the connected component of SO(3,1) with the double cover : $\mu : Spin(3,1) \to SO_{.0}(3,1) :: \mu(\pm s) = g$

So $(Cl(3,1), \mathbf{Ad} \circ \mu^{-1})$ is a linear representation of $SO_0(3,1)$ on Cl(3,1) (as a vector space) and $(R^4, \jmath \circ \mu)$ is a linear representation of Spin(3,1) on R^4 :

$\forall v \in \mathbb{R}^4 : \mathbf{Ad}(s) v = s \cdot v \cdot s^{-1} = [¡(\mu(s))] v$

4) The operator $ad : A \to L(A; A)$ on a Lie algebra is defined by $ad(X)(Y) = [X, Y]$, and if A is the Lie algebra of the Lie group G the operator $A : G \to L(A; A)$ is defined by $Ad(\exp X) = \exp(ad(X))$ (Knapp [12] p.80). One has the identity :

$g \in G, X \in A : g(\exp X)g^{-1} = \exp(Ad_g(X))$. The couple (A,Ad) is the adjoint representation of G.

A linear representation (F, ρ) of a Lie group G induces a linear representation $(F, \rho'(1))$ of its Lie algebra A :

$\forall X, Y \in A : \rho'(1)([X, Y]_A) = [\rho'(1) X, \rho'(1) Y] = (\rho'(1) X) \circ (\rho'(1) Y) - (\rho'(1) Y) \circ (\rho'(1) X)$

and $\rho'(1) X = \frac{d}{d\tau} \rho(\exp_G \tau X)|_{\tau=0}$ where exp is defined over G (as a manifold) and A. If G is a group of matrices then exp can be computed as the exponential of matrices.

Spin(3,1) being the covering group of SOl(3,1) they share the same Lie algebra o(3,1) and :

$\forall X \in o(3,1), s \in Spin(3,1) : Ad_s X = Ad_{\mu(s)} X = \exp(ad(X))$

To sum up we have the following representations of Lie groups and Lie algebra :

$SO(3,1) : (o(3,1), Ad) \to o(3,1) : (o(3,1), ad)$
$SO(3,1) : (\mathbb{R}^4, \jmath) \to o(3,1) : (\mathbb{R}^4, \jmath'(1))$
$Spin(3,1) : (\mathbb{R}^4, \jmath \circ \mu) \to o(3,1) : (\mathbb{R}^4, \jmath' \circ \mu'(1))$
$Spin(3,1) : (Cl(3,1), \mathbf{Ad}) \to o(3,1) : (Cl(3,1), \mathbf{Ad}'(1))$
$SO_0(3,1) : (Cl(3,1), \mathbf{Ad} \circ \mu^{-1})$

5) A great advantage of Clifford algebra is that vectors, groups and Lie algebra can be expressed as sum of products of vectors. This is the case for the algebra o(3,1), which is characterized by the brackets relations :

$[\overrightarrow{\kappa}_i, \overrightarrow{\kappa}_j] = \epsilon(i,j,k) \overrightarrow{\kappa}_k$
$[\overrightarrow{\kappa}_{i+3}, \overrightarrow{\kappa}_{j+3}] = -\epsilon(i,j,k) \overrightarrow{\kappa}_k$
$[\overrightarrow{\kappa}_i, \overrightarrow{\kappa}_{j+3}] = \epsilon(i,j,k) \overrightarrow{\kappa}_{k+3}$

where the indexes i,j,k run over 1,2,3 and $\epsilon(i,j,k)=0$ is two indexes are equal, and equal to the signature of (i,j,k) if not. In its standard representation the 6 following matrices are a basis for o(3,1):

$$[\widetilde{\kappa}_1]=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&-1\\0&0&1&0\end{bmatrix};[\widetilde{\kappa}_2]=\begin{bmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&-1&0&0\end{bmatrix};[\widetilde{\kappa}_3]=\begin{bmatrix}0&0&0&0\\0&0&-1&0\\0&1&0&0\\0&0&0&0\end{bmatrix}$$
$$[\widetilde{\kappa}_4]=\begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix};[\widetilde{\kappa}_5]=\begin{bmatrix}0&0&1&0\\0&0&0&0\\1&0&0&0\\0&0&0&0\end{bmatrix};[\widetilde{\kappa}_6]=\begin{bmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&0\end{bmatrix}$$

Then their image in Cl(3,1) by $\widehat{\jmath}:o(3,1)\to Cl(3,1)$ are the following:

$\widehat{\jmath}(\overrightarrow{\kappa}_1)=\frac{1}{2}\varepsilon_3\cdot\varepsilon_2;\widehat{\jmath}(\overrightarrow{\kappa}_2)=\frac{1}{2}\varepsilon_1\cdot\varepsilon_3;\widehat{\jmath}(\overrightarrow{\kappa}_3)=\frac{1}{2}\varepsilon_2\cdot\varepsilon_1;$
$\widehat{\jmath}(\overrightarrow{\kappa}_4)=\frac{1}{2}\varepsilon_0\cdot\varepsilon_1;\widehat{\jmath}(\overrightarrow{\kappa}_5)=\frac{1}{2}\varepsilon_0\cdot\varepsilon_2;\widehat{\jmath}(\overrightarrow{\kappa}_6)=\frac{1}{2}\varepsilon_0\cdot\varepsilon_3;$

One can write : $\widehat{\jmath}(\overrightarrow{\kappa}_a)=\frac{1}{2}\varepsilon_{p_a}\cdot\varepsilon_{q_a}$ with the following table :

TABLE 1 :
$$\begin{bmatrix}a&1&2&3&4&5&6\\p_a&3&1&2&0&0&0\\q_a&2&3&1&1&2&3\end{bmatrix}$$

and it is easy to check that :

$$[\widetilde{\kappa}_a]^i_j=\left(\delta^{ip_a}\delta_{jq_a}-\delta^{iq_a}\eta_{jp_a}\right)\tag{1}$$

Spin(3,1) acts on the vectors (in Cl(3,1)) by **Ad** and the result is such that :

$\forall v\in\mathbb{R}^4:\mathbf{Ad}(s)\,v=s.v.s^{-1}=\jmath\circ\mu(s)\,v$

where $\jmath\circ\mu(s)$ is expressed as a matrix $\jmath\circ\mu(s)$ belonging to the standard representation of SO(3,1).

An element of Spin(3,1) is the sum of the products of an even number of vectors with norm 1 :

$v_k=v_1\varepsilon_1+v_2\varepsilon_2+v_3\varepsilon_3+v_0\varepsilon_0, v_\alpha\in R$

$\langle v_k,v_k\rangle=v_1^2+v_2^2+v_3^2-v_0^2=1$

Direct computation gives an expression :

$S = T + Y_3\varepsilon_1\varepsilon_2 + Y_2\varepsilon_1\varepsilon_3 + Y_1\varepsilon_2\varepsilon_3 + Z_1\varepsilon_0\varepsilon_1 + Z_2\varepsilon_0\varepsilon_2 + Z_3\varepsilon_0\varepsilon_3 + U\varepsilon_0\varepsilon_1\varepsilon_2\varepsilon_3$ with real coefficients, which are not independant (there are only 6 degrees of freedom).

Let $\overrightarrow{\kappa} = \sum_a \kappa^a \overrightarrow{\kappa}_a$ be a fixed vector in o(3,1), with components κ^a in a base $\overrightarrow{\kappa}_a$, $\tau \in \mathbb{R}$ and $s(\tau) = \exp \tau \overrightarrow{\kappa}$. We have :

$\exp \tau \overrightarrow{\kappa} = T + Y_3\varepsilon_1\varepsilon_2 + Y_2\varepsilon_1\varepsilon_3 + Y_1\varepsilon_2\varepsilon_3 + Z_1\varepsilon_0\varepsilon_1 + Z_2\varepsilon_0\varepsilon_2 + Z_3\varepsilon_0\varepsilon_3 + U\varepsilon_0\varepsilon_1\varepsilon_2\varepsilon_3$ with $T(\tau\kappa), Y_i(\tau\kappa), Z_i(\tau\kappa), U(\tau\kappa)$.

differentiation in $\tau = 0$ gives :

$\frac{d}{d\tau}(\exp \tau \overrightarrow{\kappa})|_{\tau=0}$
$= \sum_a \kappa^a \frac{\partial}{\partial \kappa_a} T(1) + \kappa^a \frac{\partial}{\partial \kappa_a} Y_3(1)\varepsilon_1\varepsilon_2 + \kappa^a \frac{\partial}{\partial \kappa_a} Y_2(1)\varepsilon_1\varepsilon_3$
$+\kappa^a \frac{\partial}{\partial \kappa_a} Y_1(1)\varepsilon_2\varepsilon_3 + \kappa^a \frac{\partial}{\partial \kappa_a} Z_1(1)\varepsilon_0\varepsilon_1 + \kappa^a \frac{\partial}{\partial \kappa_a} Z_2(1)\varepsilon_0\varepsilon_2$
$+\kappa^a \frac{\partial}{\partial \kappa_a} Z_3(1)\varepsilon_0\varepsilon_3 + \kappa^a \frac{\partial}{\partial \kappa_a} U(1)\varepsilon_0\varepsilon_1\varepsilon_2\varepsilon_3$

which can be written :

$\kappa = \widehat{\jmath}(\overrightarrow{\kappa}) = t + y_3\varepsilon_1\varepsilon_2 + y_2\varepsilon_1\varepsilon_3 + y_1\varepsilon_2\varepsilon_3 + z_1\varepsilon_0\varepsilon_1 + z_2\varepsilon_0\varepsilon_2 + z_3\varepsilon_0\varepsilon_3 + u\varepsilon_0\varepsilon_1\varepsilon_2\varepsilon_3$

where :

$t = \sum_a \kappa^a \frac{\partial}{\partial \kappa_a} T(1), y_j = \sum_a \kappa^a \frac{\partial}{\partial \kappa_a} Y_j(1), z_j = \sum_a \kappa^a \frac{\partial}{\partial \kappa_a} Z_j(1),$
$u = \sum_a \kappa^a \frac{\partial}{\partial \kappa_a} U(1)$

are the components of an element of o(3,1) expressed in the base of Cl(3,1).

On the other hand the differentiation of :

$\mathbf{Ad}(\exp \tau \overrightarrow{\kappa}) v = (\exp \tau \overrightarrow{\kappa}) \cdot v \cdot (\exp \tau \overrightarrow{\kappa})^{-1} = [\jmath \circ \mu (\exp \tau \overrightarrow{\kappa})] v$

gives : $(\mathbf{Ad}'(1)\overrightarrow{\kappa}) v = \kappa \cdot v - v \cdot \kappa = [\jmath'(1)\mu'(1)\overrightarrow{\kappa}](v) = [\widetilde{\kappa}] v$

where $[\widetilde{\kappa}] = [\jmath'(1)\mu'(1)\overrightarrow{\kappa}]$ is the matrix representing $\overrightarrow{\kappa}$ in the standard representation.

Expressing κ as above and v as $v = v_1\varepsilon_1 + v_2\varepsilon_2 + v_3\varepsilon_3 + v_0\varepsilon_0$ we get :

$\kappa \cdot v - v \cdot \kappa = \sum_{kl} [\widetilde{\kappa}]^k_l v_l \varepsilon_k$
$= (-2)\{(-v_0 z_1 - v_3 y_2 - v_2 y_3)\varepsilon_1 + (v_1 y_3 - v_3 y_1 - v_0 z_2)\varepsilon_2$
$+ (v_1 y_2 + v_2 y_1 - v_0 z_3)\varepsilon_3 - (v_1 z_1 + v_2 z_2 - v_3 z_3)\varepsilon_0\}$
$+ (2) u ((v_0)\varepsilon_1\varepsilon_2\varepsilon_3 + (v_3)\varepsilon_0\varepsilon_1\varepsilon_2 - (v_2)\varepsilon_0\varepsilon_1\varepsilon_3 + (v_1)\varepsilon_0\varepsilon_2\varepsilon_3)$
$\Rightarrow u = 0$

$$\kappa \cdot v - v \cdot \kappa = (-2)\begin{bmatrix} 0 & -z_1 & -z_2 & -z_3 \\ -z_1 & 0 & -y_3 & -y_2 \\ -z_2 & y_3 & 0 & -y_1 \\ -z_3 & y_2 & y_1 & 0 \end{bmatrix}\begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix} = \sum_a \kappa^a [\widetilde{\kappa}_a]^k_l v_l \varepsilon_k$$

where $[\widetilde{\kappa}_a] = [\jmath'(1)\mu'(1)\overrightarrow{\kappa}_a]$
$\sum_a \kappa^a [\widetilde{\kappa}_a] = (-2)(-z_1[\widetilde{\kappa}_4] - z_2[\widetilde{\kappa}_5] - z_3[\widetilde{\kappa}_6] + y_1[\widetilde{\kappa}_1] - y_2[\widetilde{\kappa}_2] + y_3[\widetilde{\kappa}_3])$

$= \sum_a \kappa^a \left[\jmath'(1) \mu'(1) \overrightarrow{\kappa}_a \right]$

By identifying with $\kappa = \sum_a \kappa^a \kappa_a = y_3 \varepsilon_1 \varepsilon_2 + y_2 \varepsilon_1 \varepsilon_3 + y_1 \varepsilon_2 \varepsilon_3 + z_1 \varepsilon_0 \varepsilon_1 + z_2 \varepsilon_0 \varepsilon_2 + z_3 \varepsilon_0 \varepsilon_3$ we get the expected expression of a basis of a 6-dimensional space vector in Cl(3,1) which defines the image of o(3,1).■

6) These results can be extended to the Cl(4,C) Clifford algebra and its Cl_c(3,1) subalgebra. The function $\Upsilon : Cl(3,1) \rightarrow Cl(4,C)$ is an isomorphism, so Spin(3,1) has an image $Spin_c(3,1)$ which is a subgroup of Spin(4,C) isomorphic to Spin(3,1). There is a similar construction with Cl(1,3) with the function $\Upsilon' : Cl(1,3) \rightarrow Cl(4,C)$ which gives a group Spin(1,3) isomorphic to Spin(3,1), and we have $Spin_c(3,1) \neq Spin_c(1,3)$ but the two are also isomorphic.

2.1.3 The spin bundle

1) Most of the geometric transformations can be described by operations in Clifford algebra. Reflections on a plane of normal vector u is just $\alpha(u) \cdot v \cdot u^{-1}$. Rotations can be expressed as the product of reflexions, and indeed this fact is at the foundation of the Spin group through the map **Ad**. Clifford algebras give a deep insight at the role of Spin groups to express rotations in physics. Rotation of a body around its center mass can be expressed in the local frame by the axis j and the angular speed ω. For an observer who is at rest with the body the couples (j, ω) and $(-j, -\omega)$ represent the same rotation. But if we think to a network of observers, who have to coordinate their data and the two representations are equivalent but not identical : measuring the rotation implies moving the local frame (posibly in "the time dimension") and the choice between (j, ω) and $(-j, -\omega)$ matters : the observer must choose the orientation of the axis which will be parallel transported. This issue is related to the fact that the group SO(3) is not simply connected (as well as SO(3,1)) (see Penrose [20] 11.3 for a nice experiment on the subject) . The full description of the spatial rotation of a body, in its local environment, needs to be done in the Spin(3) group, which is the universal covering group of SO(3) (as Spin(3,1) is to $SO_0(3,1)$). For each $g \in SO_0(3,1)$there are 2 members of Spin(3,1) $\pm s$ which differ by their sign.

2) The construction of a Clifford bundle over a manifold endowed with a metric occurs naturally. At each point m of the manifold where there is an orthonormal basis $\partial_i(m)$ one defines the map : $\varphi_c : M \times Cl(3,1) \rightarrow Cl(M)$

by identifying the generators : $\partial_i(m) = \varphi_c(m, \varepsilon_i)$. Cl(M) is an algebra bundle, and the associated vector bundle G_M is identified with the vector space part of Cl(M). Spin(3,1) acts upon Cl(M) by $\mathbf{Ad}$, which, for the vectors, results in a change of othonormal base : $s \times \varphi_c(m, \varepsilon_i) = \varphi_c(m, \mathbf{Ad}_s \varepsilon_i) = \varphi_c(m, \mu(s)\varepsilon_i) = [\jmath \circ \mu(s)]_i^j \partial_j$

3) The construction of a principal fiber bundle S_M modelled over Spin(3,1) is more subtle. Starting from the principal fiber bundle O_M it can can be done if there is a spin structure : a map $\Xi : S_M \to O_M$ such that $\Xi(\varphi_S(m,s)) = \varphi_O(m, \mu(s))$ such that $\Xi(\varphi_S(m,s).s') = \Xi(\varphi_S(m,s))\mu(s')$ meaning that one can choose one of the two members +s or -s corresponding to an element of SO(3,1) in a consistent and continuous manner over M. There are topological obstructions to the existence and unicity of spin structures over a manifold (Giachetta [5] 7.2, Svetlichny [25]) but, as our system covers a limited, not too exotic, region of M, we can assume that the procedure can be implemented to get the principal fiber bundle $S_M = (S_M, \Omega, \pi_S, Spin(3,1))$ based upon Ω with the projection $\pi_S : S_M \to \Omega$, local trivialization charts $\varphi_S : \Omega \times Spin(3,1) \to S_M$.

The associated vector bundle G_M can then be extended to an associated vector bundle $S_M \times_{Spin(3,1)} R^4$ which is still denoted G_M for simplicity. The sections $\partial_i(m)$ are associated with the identity element 1_S of Spin(3,1).

2.1.4 Representation of Clifford algebra

1) There are two ways to deal with Clifford algebra. Either directly, using the operations of the algebra. Or through a linear representation of the algebra, using the fact that Clifford algebra are isomorphic to matrices algebra. For the computation side the choice is mainly a matter of personal preference. But we have here several issues.

i) The theory of fields, with connexions and variational calculus is, so far, not well suited to algebra per se.

ii) As usual it can be useful to add more physical content to the model. It is easier to do it by some specification of the vector space over which the algebra is represented (we will see this with chirality).

iii) There are strong evidences which suggest the use of a complex structure, besides the fact that most of the mathematical tools require it.

iv) Cl(1,3) and Cl(3,1) are not isomorphic : Cl(3,1) is isomorphic to the algebra of 4x4 matrices over $\mathbb{R}$, Cl(1,3) to the 2x2 matrices over the

quaternions, so if we stay to the pure algebra we need to choose the signature.

For all these motives the choice done here is to use *a representation of the Cl(4,C) algebra*, the complexified of both Cl(1,3) and Cl(3,1), which is isomorphic to the 4x4 matrices over $\mathbb{C}$. This is an irreducible representation of Cl(4,C).

2) So the kinematic state of a particle is assumed to be a vector ϕ of a 4 dimensional complex space F, so far unspecified, and (F,ρ) is a representation of Cl(4,C), meaning that :

$\rho : Cl(4,C) \to L(F,F)$ ::

$\forall w, w' \in Cl(4,C), \alpha, \beta \in \mathbb{C} : \rho\left(w \cdot w'\right) = \rho\left(w\right) \circ \rho\left(w'\right)$

$\rho\left(\alpha w + \beta w'\right) = \alpha\rho\left(w\right) + \beta\rho\left(w'\right).$

By restriction :

(F,ρ) is a representation of the algebra $Cl_c\left(3,1\right), Cl_c\left(1,3\right)$, and of the groups $Spin(4,C), Spin_c\left(3,1\right), Spin_c\left(1,3\right),$

$\left(F, \rho \circ \Upsilon\right), \left(F, \rho \circ \Upsilon'\right)$ are representations of the algebra Cl(3,1) and Cl(1,3)

$\left(F, \rho \circ \Upsilon|_{Spin(3,1)}\right), \left(F, \rho \circ \Upsilon'|_{Spin(1,3)}\right)$ are complex representations of the groups Spin(3,1), Spin(1,3)

$\rho \circ \Upsilon'(s)|_{s=1} : o(3,1) \to \hom(F;F)$ is a complex representation of the Lie algebra o(3,1)

3) F being 4 dimensional, in a basis denoted $\left(e_j\right)_{j=1}^4$ $\rho\left(w\right)$ is a 4x4 matrice over $\mathbb{C}$ fully determined by fixing the matrices $\gamma_k = \rho\left(\varepsilon_k\right), k = 0,..3$ which shall meet the relations :

$$\varepsilon_j \in \mathbb{C}^4 : \gamma_j = \rho\left(\varepsilon_j\right); \gamma_i \times \gamma_j + \gamma_j \times \gamma_i = \mathbf{2}\delta_{ij}\mathbf{I}_4 \tag{2}$$

These matrices are defined up to conjugation by a matrix. It is always possible to choose hermitian matrices, so we assume that : $\gamma_k = \gamma_k^*$. As $\gamma_k\gamma_k = I$ they are also unitary.

Remark : these γ matrices, which will be specified later, are not the usual "Dirac matrices", which correspond to a representation of Cl(3,1) (or Cl(1,3)).

4) To the matrices $\widehat{\jmath}(\overrightarrow{\kappa}_a) = \frac{1}{2}\varepsilon_{p_a} \cdot \varepsilon_{q_a}$ representing elements of o(3,1) in Cl(3,1) correspond the matrices :

$[\kappa_a] = \left(\rho \circ \Upsilon\right)'(1)\left(\widehat{\jmath}(\overrightarrow{\kappa}_a)\right) = \left(\rho \circ \Upsilon\right)'(1)\left(\frac{1}{2}\varepsilon_{p_a} \cdot \varepsilon_{q_a}\right)$

Υ is a linear morphism $Cl(3,1) \to Cl(4,C)$:
$\Upsilon'(1)\left(\frac{1}{2}\varepsilon_{p_a}\cdot\varepsilon_{q_a}\right) = \frac{1}{2}\Upsilon\left(\varepsilon_{p_a}\right)\cdot\Upsilon\left(\varepsilon_{q_a}\right)$
ρ is a linear morphism $Cl(4,C) \to L(F;F)$:
$\rho'(1)\left(\frac{1}{2}\Upsilon\left(\varepsilon_{p_a}\right)\cdot\Upsilon\left(\varepsilon_{q_a}\right)\right) = \rho\left(\frac{1}{2}\Upsilon\left(\varepsilon_{p_a}\right)\cdot\Upsilon\left(\varepsilon_{q_a}\right)\right) = \frac{1}{2}\rho\left(\Upsilon\left(\varepsilon_{p_a}\right)\right)\rho\left(\Upsilon\left(\varepsilon_{q_a}\right)\right)$
with $\Upsilon(\varepsilon_0) = i\varepsilon_0, \Upsilon(\varepsilon_k) = \varepsilon_k : k = 1,2,3$ we get :

$$[\kappa_1]=\frac{1}{2}\gamma_3\gamma_2; [\kappa_2]=\frac{1}{2}\gamma_1\gamma_3; [\kappa_3]=\frac{1}{2}\gamma_2\gamma_1; \tag{3a}$$

$$[\kappa_4]=\frac{1}{2}\mathbf{i}\gamma_0\gamma_1; [\kappa_5]=\frac{1}{2}\mathbf{i}\gamma_0\gamma_2; [\kappa_6]=\frac{1}{2}\mathbf{i}\gamma_0\gamma_3 \tag{3b}$$

Notice that the matrices in the standard representation $(\mathbb{R}^4, \jmath)$ are denoted with a tilde : $[\widetilde{\kappa}_a]$

5) An associated vector bundle modelled over F : $S_M \times_{Spin(3,1)} F$ can be defined if there is a spin structure, and the kinematic state of a particle is a represented by a vector of this vector bundle. Spin(3,1) acts over the bundle by $\rho \circ \Upsilon$.

6) (F, ρ) is a representation of Cl(4,C) so does not depend on the signature which is used, and the γ matrices are the same. But all the items computed through Υ have different value. The basic rule is : $\Upsilon'(\varepsilon_j) = -i\eta^{jj}\Upsilon(\varepsilon_j)$ so
$[\kappa'_a] = \frac{1}{2}\rho\left(\Upsilon'\left(\varepsilon_{p_a}\right)\right)\rho\Upsilon'\left(\varepsilon_{q_a}\right) = -\eta^{p_a p_a}[\kappa_a]$
$a < 4 : [\kappa'_a] = -[\kappa_a], a > 3 : [\kappa'_a] = [\kappa_a]$

Kinematic states are supposed to be physical objects represented in F. If a kinematic state is represented by a vector $\phi \in F$ with the signature (- + + +), it would be represented by $\phi' = \sum_{j=1}^{4} -i\eta^{jj}\phi^j e_j$ with the signature (+ - - -).

2.2 Physical characteristics

2.2.1 Definitions

1) The interaction beween force fields and particles depend upon some features specific to each kind of particle, such as electric charge. We know that these features are quantized, but their quantization in the classical picture, and indeed their exact value in any picture, should be the result of the model, so I assume that the physical characteristics of a particle are part of its state,

and can take a priori a continuum of values, and therefore we do not need to distinguish several kind of particles. We know that these characteristics can be described in the linear representations of some groups. I will not be too specific about the nature of the forces fields, and make the following assumptions :

i) The physical characteristics of a particle can be modelled by a vector σ of a complex vector space W

ii) The couple (W, χ) is a linear representation of some group U

iii) U is a connected compact Lie group

The last assumption is reasonable and it brings us nice properties : there is a scalar product over W, such that the representation is unitary, and any unitary representation is the sum of irreducible unitary finite dimensional representations which are orthogonal. Such irreducible representations are natural candidates to describe different families of particles. So I will assume that iv) the representation (W, χ) is unitary and W is m-dimensional, but not necessarily irreducible.

Notation 2 *:*

The identity in U is denoted : 1_U or 1

The Lie algebra of U is denoted T_1U . It is a real vector space with basis $\left(\overrightarrow{\theta}_a\right)_{a=1}^m$ so $T_1U = \left\{\overrightarrow{\theta} = \sum_{a=1}^m \theta^a \overrightarrow{\theta}_a, \theta^a \in \mathbb{R}\right\}$. Its structure coefficients C_{bc}^a are real numbers. U being a compact Lie group the exponential map is on-to and $u \in U$ can be written $u = \exp\left(\sum_{a=1}^m \theta^a \overrightarrow{\theta}_a\right)$ for some components θ^a.

The hermitian scalar product in W is denoted $\langle \lambda\sigma_1, \sigma_2\rangle = \overline{\lambda}\langle \sigma_1, \sigma_2\rangle$ with an orthonormal basis $(f_i)_{i=1}^m$ and the matrices $[\chi(u)]$ are unitary : $[\chi(u)][\chi(u)]^* = I_m$

$(W, \chi'(1))$ is a linear representation of the Lie algebra T_1U and for each $\overrightarrow{\theta} \in T_1U$ the matrix $[\theta] = \chi'(1)\overrightarrow{\theta}$ in the basis $(f_i)_{i=1}^m$ is antihermitian : $[\theta]^* = -[\theta]$. The matrices corresponding to the basis $\overrightarrow{\theta}_a$ are denoted $[\theta_a]$. There are the relations :

$\forall \overrightarrow{\theta} \in T_1U, \tau \in \mathbb{R} : [\theta] = \left[\chi'(1)\overrightarrow{\theta}\right] = \frac{d}{d\tau}\left[\chi\left(\exp\tau\overrightarrow{\theta}\right)\right]|_{\tau=0}$

and $\left[\chi\left(\exp\tau\overrightarrow{\theta}\right)\right] = \exp\left[\tau\overrightarrow{\theta}\right]$ can be computed as exponential of matrices.

3) We will need the complexified $T_1^cU = T_1U \oplus iT_1U$ of the Lie algebra, with the same basis (but complex components) and the complex extension of the bracket. The representation $(W, \chi'(1))$ can be extended to a representation $(W, \chi'_c(1))$ of T_1^cU by making $\chi'(1)$ a complex linear map : $\chi'_c(1)\, i\overrightarrow{\theta} = i\chi'(1)\overrightarrow{\theta}$. But the matrices $[\theta]$ are antihermitian only if $\overrightarrow{\theta} \in T_1U$.

U being a connected compact Lie group is isomorphic to a linear group (of matrices) closed in GL(m) and there is a complex analytic group of matrices U_c which admits T_1^cU as Lie algebra. U_c is not unique, but all these groups are isomorphic (Knapp [11] 7.12). I will assume that (W, χ) can be extended to a linear representation (not necessarily unitary) of U_c. It can be done if U is simply connected or admits a Cartan decomposition. This is the case for the usual groups SU(N), which have su(N) as Lie algebra (which is a real algebra) and SL(N,C) and sl(N,C) as complexified structures.

2.2.2 The bundle of particles states

The state of a particle is described in the tensor product $E = F \otimes W$. It is a tensor $\psi = \sum_{i=1}^{4} \sum_{j=1}^{m} \psi^{ij} e_i \otimes f_j$.

We need a geometric structure over M bearing this representation.

1) We have the principal fiber bundle S_M . We assume that there is a principal bundle U_M over M modelled over the group U. These two fiber bundles are manifolds, so we can, starting from one or the other, define the fiber bundle Q_M, with base M, typical fiber Spin(3,1)xU and trivializations :

$q = \varphi_Q(m, (s, u)) \times (s', v) \to \varphi_Q(m, (ss', uv))$

$\varphi_Q(m, (1_{Spin}, u)), \varphi_Q(m, (s, 1_G))$ are trivializations of U_M, S_M and the bundles S_M, U_M can be seen as sub-bundles of the principal bundle Q_M.

We denote the section $q(m) = \varphi_q(m, 1_{Spin} \times 1_G) \in \Lambda_0 Q_M$

2) We define the action of Cl(3,1)xU on FxW by :

$(\phi, \sigma) \times (w, u) \to (\rho \circ \Upsilon(w)\phi, \chi(u)\sigma)$

For any (w,u) this action is a bilinear map over FxW. So by the universal property of the tensor product there is one unique linear map :

$\vartheta : Cl(3,1) \otimes U \to L(F \otimes W; F \otimes W)$

such that :

$\forall (\sigma, \phi) \in F \times W : \vartheta(w, u)(\phi \otimes \sigma) = (\rho \circ \Upsilon(w)\phi) \otimes (\chi(u)\sigma)$

The action of ϑ restricted to Spin(3,1)xU is an action of the direct product and $(F \otimes W, \vartheta)$ is a linear representation of the group Spin(3,1)xU.

3) From there the **vector bundle** E_M **associated to** Q_M is defined through the action :

$(q, \psi) \times (s, u) \rightarrow \left(q (s, u)^{-1}, \vartheta (s, u) (\psi)\right)$

and the equivalence relation : $(q, \psi) \simeq \left(q (s, u)^{-1}, \vartheta (s, u) (\psi)\right)$

We denote the sections i=1..4;j=1..m :

$e_i (m) \otimes f_j (m) = (\varphi_q (m, 1_{Spin} \times 1_U), e_i \otimes f_j)$

$\simeq \left(q (m) (s, u)^{-1}, (\rho \circ \Upsilon (s) e_i) \otimes (\chi (u) f_j)\right)$

which define a local basis of the fiber $E_M (m)$.

A section of E_M is a map :$\Omega \rightarrow E_M : \psi (m) = \psi^{ij} (m) e_i (m) \otimes f_i (m)$.

4) So far the choice of the vector spaces F and W is open, but we need some procedure to measure the components ψ^{ij}, that is a way for an observer to define the vectors $(e_i (m), f_j (m))$.

The basis $e_i (m)$ transform according to the same rules as $\partial_i (m)$ and has clearly a geometric meaning. So we assume there is some procedure to relate the two bases. The basis $f_j (m)$ is related to the action of the force fields on the particles, and should be defined form the trajectories of test particles. The existence of the principal fiber bundle U_M needs some procedure to compare the measures done by observers in different locations. This issue will be addressed below.

2.2.3 Spinor and Clifford algebra

It is useful to link the present model to the usual "spinors" used in quantum physics to describe spinning particles. The situation of quantum physics is indeed a bit complicated.

1) As was said previously, in a local field theory any physical quantity which is expressed as a tensor must be a section of a vector bundle, associated with a principal bundle of the world manifold and modelled over a vector space which is a representation of the gauge group. There are well known, but fairly technical, methods to find all linear representations of a group. Usually the solution is a representation of the covering group, meaning a multi-valued

representation. The double cover of $SO_0(3,1)$ is Spin(3,1), which is isomorphic to SL(2,C). Its representations are the direct product of the usual "spin" representations of SO(3), and are indexed by 2 integers or half integers (see Tung [28] and Knapp [11]). So in the relativistic picture (special or general) the physical vectors (whenever they are supposed to represent a geometric quantity) belong to a vector space which is a representation of SL(2,C), assuming that there is a "spin structure". The "Weyl spinors" are $\mathbb{C}^2$ vectors corresponding to one of the two non equivalent representations (1/2,0), (0,1/2). The "Dirac spinors" are $\mathbb{C}^4$ vectors corresponding to the $(0,1/2) \oplus (1/2,0)$ representation, which is the 4 complex dimensional representation of the Spin(3,1) group.

Notice that this prescription follows from the principles of locality and relativity, and stands for classical as well as for quantum models.

2) But in quantum mechanics there is also the Wigner theorem, which states that, whenever there is some gauge group, observables must be expressed in a projective representation of this group. It is possible to get rid of the phase factor, and go for a regular representation, if the Lie group is semi-simple and simply connected (see Weinberg [30] I.2). This second condition, infortunately, is not met by SO(3) or SO(3,1). There are some ways around this issue, coming eventually to a representation of the covering group, which is what one gets anyway, and impose a "super-selection" rule between the 2 states. The problem is that the only unitary representations of SL(2,C) (and SO(3,1)) are infinite dimensional (see Knapp [11]).

3) In Special Relativity the gauge group can be extended to the Poincaré group. There is still no finite dimensional unitary representation but, if one fixes one 4-vector $\overrightarrow{p} \neq 0$ the irreducible unitary representation of SO(3) (if $\|p\| \neq 0$) of SO(2) (if $\|p\| = 0$) are also irreducible unitary representation of the subgoup of the Poincaré group leaving $\overrightarrow{p}$ invariant. These representations are labelled by $\overrightarrow{p}$ and the spin s (for fermions) or helicity (for massless particles). They are infinite dimensional unitary representations over an Hilbert space of functions of p (and labelled by s). Their Fourier transform gives back functions of the coordinates, which, for the massive particles, can be expressed as functions of space-time coordinates valued in one of the finite-dimensional representation of SL(2,C).These relativistic wave functions can be seen as plane-waves which combine to give the actual particles, through a

process of anhiliation and creation. They are labelled by both the representation (s) and by other quantum numbers which characterize the particle. In quantum theory of fields observables are localized operators acting on these wave-functions.

3 FORCE FIELDS

3.1 Principles

1) Force fields interact with particles (remind that here particles are matter particles) : they change their trajectories (and possibly their physical characteristics) and conversely the particles change the strength of the fields. Moreover the force fields are defined all over the universe, and propagate without staying the same, even is there is no source : they interact which each others. In a local field theory these interactions are purely local : they are determined by the value of the fields and the states of the particles which are present at the same location of the space-time.

2) The action of a force field on a particle depends on and changes the state $\psi(m)$ and the velocity of the particle. The simplest assumption is that this action is linear, and can be modelled by some map over $\Lambda_0 E_M$. It depends also on the trajectory of the particle : indeed particles always move on their world line, so the value of the field that the particle meets is changing and by the same mechanism the presence of the particle changes the value of the field. If we keep the assumption of linearity the action of the field is reasonably modelled by a 1-form over M, valued in E_M : that is by a connection. Gauge equivariance implies equivariance of the connection, which is therefore a connection associated to a principal connection on the principal fiber bundle $Q(m)$.

3) According to General Relativity inertial forces are equivalent to gravitational forces and related to the curvature of space-time. As far as we know they change the trajectories and the kinematic state of particles, but not their physical characteristics. So the gravitational field will be modelled as a principal connection $\mathbf{G}$ over S_M acting on the kinematic part of ψ (in F) and on the velocity of the particle. The "other field forces" will be modelled

as a principal connection **A** over U_M acting on the other part (in W) of the state and on the velocity.

3.2 Gravitation

There are different approaches to the modellization of gravitation, related to the two different pictures of the geometry of the universe.

1)à The traditional way stems from the description of M as a manifold endowed with a metric g, and so g is the central piece. An affine connection (also called a "world connection") can be seen as a linear connection on the tangent bundle, which is no other than the vector bundle TM associated to the principal bundle modelled on GL(4). It induces a covariant derivative $\widetilde{\nabla}$ acting on the sections of TM (the vector fields) characterized by the Christoffel coefficients $\Gamma^{\alpha}_{\beta\gamma}$ in an holonomic basis, and an exterior covariant derivative $\widetilde{\nabla}_e$ acting on the forms over TM*, characterized by the Riemann tensor R and the torsion. So far there is nothing which requires a metric. The connection is metric if it preserves the scalar product, which is equivalent to the condition : $\widetilde{\nabla} g = 0 \Leftrightarrow \Gamma^{\varepsilon}_{\alpha\beta} g_{\varepsilon\gamma} + g_{\beta\varepsilon} \Gamma^{\varepsilon}_{\alpha\gamma} = \partial_\alpha g_{\beta\gamma}$.

It is symmetric if the torsion is null, which is equivalent to : $\Gamma^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\gamma\beta}$. There is a unique affine connection which meets these two conditions : the Lévy-Civita connection whose Christoffel coefficients are a function of the first order partial derivatives of g.

From there if one takes g as the key variable, and imposes that Γ is defined by some operator $g \to \Gamma$ which cannot depend on the choice of an holonomic basis (it is a "natural operator" in the categories parlance) the unique first order solution is the Lévy-Civita connection (Kolar [14] 52.3). A theorem by Utiyama says that if the lagrangian depends only on the first derivatives of g it must factorize through the scalar curvature R. Additional algebraic conditions then lead for the lagrangian to the specification : $L_G \varpi_4 = a\,(R + \Lambda) \sqrt{|\det g|} \varpi_0$ (with a cosmological constant Λ). So the problem is fairly delimited and we are in the usual framework of General Relativity.

If no relation is imposed a priori between g and Γ one has the so-called Einstein-Cartan models (Trautman [27]). The compatibility between the connection and the metric is an external constraint, and generally the connection is not torsionfree.

In both cases the variational calculus can be done in an holonomic basis (Soper [24]) or in an non-holonomic orthonormal basis. The latter method (by "tetrads") has numerous variants (Wald [29]) but the use of an orthonormal basis is mainly a way to simplify calculations which are always difficult.

2) In the alternate approach to the geometry let us assume that there is a principal connection **G** on the principal bundle S_M represented by its connection 1-form $\widehat{G} : S_M \to \Lambda_1 (TM^*; o(3,1))$ and its potential : $G(m) = \widehat{G}(\varphi_S(m,1))$. Under a local jauge transformation $\varphi_S(m,1) \to \varphi_S\left(m, h(m)^{-1}\right)$ (h varies with m) $\widehat{G}$ changes as :

$\widehat{G}(\varphi_S(m, h^{-1}))$
$= Ad_h \widehat{G}(\varphi_S(m,1)) + L'_h(h^{-1})(h^{-1}(m))'$
$= Ad_h \widehat{G}(\varphi_S(m,1)) - R'_{h^{-1}}(h)(h(m))'$

This connection induces a covariant derivative $\widehat{\nabla}$ over the associated vector bundle G_M. The covariant derivative of a section $V = \sum_k V^k \partial_k$ of G_M is the 1-form :

$\widehat{\nabla} V = \left(\partial_\alpha V^i + (\jmath'(1) G(m) V)^i\right) \partial_i \otimes dx^\alpha$
$= \left(\partial_\alpha V^i + \left(\sum_{a=1}^6 G_\alpha^a [\widetilde{\kappa}_a]^i_j\right) V^j\right) \partial_i \otimes dx^\alpha$

a) There is a one-one correspondance between principal connections over G_M and affine connections over TM.

Indeed the section V is a vector field $V = \sum_k V^k \partial_k = \sum_\alpha v^\alpha \partial_\alpha$ with $v^\alpha = \sum_k V^k O^\alpha_k$ and equating both derivatives :

$\widehat{\nabla} V = \left(\partial_\alpha V^i + \left(\sum_{a=1}^6 G_\alpha^a [\widetilde{\kappa}_a]^i_j\right) V^j\right) \partial_i \otimes dx^\alpha$
$\widetilde{\nabla} v = \left(\partial_\alpha v^\gamma + \Gamma^\gamma_{\alpha\beta} V^j\right) \partial_\gamma \otimes dx^\alpha$
$\widehat{\nabla} V = \widetilde{\nabla} v$ gives : $\left[\widetilde{G}_\alpha\right]^j_i = \sum_{a=1}^6 G_\alpha^a [\widetilde{\kappa}_a]^j_i = O'^j_\beta \partial_\alpha O^\beta_i + \Gamma^\beta_{\alpha\gamma} O'^j_\beta O^\gamma_i$ which can be written in matrix notation with $[\Gamma_\alpha] = [\Gamma_\alpha]^\gamma_\beta$

$$[\Gamma_\alpha] = \left([O]\left[\widetilde{G}_\alpha\right] - [\partial_\alpha O]\right)[O'] \Leftrightarrow \left[\widetilde{G}_\alpha\right] = [O']\left([\Gamma_\alpha][O] + [\partial_\alpha O]\right) \quad (4)$$

It is easy to check that conversely an affine connection defines uniquely a potential, and from there a principal connection : in the gauge transformation : $\partial_j \to \widehat{\partial}_j = S^k_j \partial_k, [S] \in SO(3,1)$ we have :

$[O] \to \left[\widehat{O}\right] = [O][S]$

$$\left[\widetilde{G}_\alpha\right] \to \widehat{\left[\widetilde{G}_\alpha\right]}$$
$$= \left[\widehat{O}'\right]\left([\Gamma_\alpha]\left[\widehat{O}\right] + \left[\partial_\alpha \widehat{O}\right]\right)$$
$$= [S^{-1}]\left[\widetilde{G}_\alpha\right][S] - [S^{-1}][O'][\partial_\alpha O][S] + [S^{-1}][O'][\partial_\alpha O][S] + [S^{-1}][O'][O][\partial_\alpha S]$$
$$= Ad_{S^{-1}}\left[\widetilde{G}_\alpha\right] + [S^{-1}][\partial_\alpha S] \blacksquare$$

b) A principal connection **G** is metric if the corresponding affine connection is metric. The necessary and sufficient condition is that : $g_{\varepsilon\gamma}\Gamma^\varepsilon_{\alpha\beta} + g_{\beta\varepsilon}\Gamma^\varepsilon_{\alpha\gamma} = \partial_\alpha g_{\beta\gamma} \Leftrightarrow ([g][\Gamma_\alpha])^t + [g][\Gamma_\alpha] = \partial_\alpha [g]$. Let us show that it is met if $[\Gamma_\alpha] = \left([O]\left[\widetilde{G}_\alpha\right] - [\partial_\alpha O]\right)[O']$ and $[g] = [O']^t[\eta][O']$

$$([g][\Gamma_\alpha])^t + [g][\Gamma_\alpha]$$
$$= \left(\left([O]\left[\widetilde{G}_\alpha\right] - [\partial_\alpha O]\right)[O']\right)^t ([O']^t[\eta][O'])^t + [O']^t[\eta][O']\left([O]\left[\widetilde{G}_\alpha\right] - [\partial_\alpha O]\right)[O']$$
$$= [O']^t\left[\widetilde{G}_\alpha\right]^t[\eta][O'] - [O']^t[\partial_\alpha O]^t[O']^t[\eta][O'] + [O']^t[\eta]\left[\widetilde{G}_\alpha\right][O'] - [O']^t[\eta][O'][\partial_\alpha O][O']$$
$$= [O']^t\left(\left[\widetilde{G}_\alpha\right]^t[\eta] - [\partial_\alpha O]^t[O']^t[\eta] + [\eta]\left[\widetilde{G}_\alpha\right] - [\eta][O'][\partial_\alpha O]\right)[O']$$
$$= -[O']^t\left([\partial_\alpha O]^t[O']^t[\eta] + [\eta][O'][\partial_\alpha O]\right)[O']$$

where we used the fact that $\left[\widetilde{G}_\alpha\right] \in o(3,1) \Leftrightarrow \left[\widetilde{G}_\alpha\right]^t[\eta] + [\eta]\left[\widetilde{G}_\alpha\right] = 0$

On the other hand we have :

$$\partial_\alpha [g]$$
$$= [\partial_\alpha O']^t[\eta][O'] + [O']^t[\eta][\partial_\alpha O']$$
$$= -[O']^t[\partial_\alpha O]^t[O']^t[\eta][O'] - [O']^t[\eta][O'][\partial_\alpha O][O']$$
$$= -[O']^t\left([\partial_\alpha O]^t[O']^t[\eta] + [\eta][O'][\partial_\alpha O]\right)[O'] \blacksquare$$

c) A principal connection **G** is symmetric is the corresponding affine connection is symmetric. Which reads :

$$\Gamma^\gamma_{\alpha\beta} = \Gamma^\gamma_{\beta\alpha} \Leftrightarrow [O]^\gamma_i\left[\widetilde{G}_\alpha\right]^i_j[O']^j_\beta - [\partial_\alpha O]^\gamma_i[O']^i_\beta = [O]^\gamma_i\left[\widetilde{G}_\beta\right]^i_j[O']^j_\alpha - [\partial_\beta O]^\gamma_i[O']^i_\alpha$$
$$[O]^\gamma_i\left(\left[\widetilde{G}_\alpha\right]^i_j[O']^j_\beta - \left[\widetilde{G}_\beta\right]^i_j[O']^j_\alpha\right) = [\partial_\alpha O]^\gamma_i[O']^i_\beta - [\partial_\beta O]^\gamma_i[O']^i_\alpha$$
$$\left[\widetilde{G}_\alpha\right]^k_j[O']^j_\beta - \left[\widetilde{G}_\beta\right]^k_j[O']^j_\alpha$$
$$= \sum\nolimits_\gamma -[O']^k_\gamma[O]^\gamma_i[\partial_\alpha O']^i_\beta + [O']^i_\alpha[O]^\gamma_i[\partial_\beta O']^k_\gamma$$

$= -\left[\partial_\alpha O'\right]^k_\beta + \left[\partial_\beta O'\right]^k_\alpha$

$$\forall \alpha, \beta, k : \left[\widetilde{G}_\alpha\right]^k_j [O']^j_\beta - \left[\widetilde{G}_\beta\right]^k_j [O']^j_\alpha = \left[\partial_\beta O'\right]^k_\alpha - \left[\partial_\alpha O'\right]^k_\beta \tag{5}$$

3) In a consistent theory of fields the connection must be metric, to guarantee that the scalar product is preserved along a geodesic. But the condition that it is torsionfree is less obvious. There is no experimental evidence on this issue (which could be a difficult one) and it seems better to keep the option open. Moreover the alternate approach, starting from a principal fiber bundle, and orthonormal basis, leads logically to put the connection itself as the key variable, and to deduce the metric from the orthonormal frames. Indeed should the metric be measured, it could be done through the relation :

$$\mathbf{g}^{\alpha\beta} = \sum_{jk} \eta^{jk} \mathbf{O}^\alpha_j \mathbf{O}^\beta_k \Leftrightarrow \mathbf{g}_{\alpha\beta} = \sum_{jk} \eta_{jk} \mathbf{O}'^j_\alpha \mathbf{O}'^k_\beta \tag{6}$$

So we will keep the connection $\mathbf{G}$ and the matrix $[O']$ as key variables, the metric g being a byproduct given by the relation above. It is clear that O' is determined within a matrix of SO(3,1) : the number of degrees of freedom with g is 6 and 16 with O', which leaves 10 degrees of freedom to fix a gauge suiting the problem. It is one of the main advantage of the tetrad method, and actually the chart which has been built previously already pre-empted such a choice. With these assumptions the connection is metric, but not necessarily symmetric.

4) I take the opportunity to introduce here some conventions and notations.

a) There is the irritating issue of the conventions about exterior product and antisymmetric tensor products. For a clear definition of the algebras of symmetric and antisymmetric tensors see Knapp ([12] A).

Here I use the following :

- I denote by $(\alpha_1, ..\alpha_r)$ any set of r indexes (taken in the pertinent set), and by $\{\alpha_1, ..\alpha_r\}$ the set of r ordered indexes : $\alpha_1 < \alpha_2 .. < \alpha_r$, by $\epsilon(\alpha_1, ..\alpha_r)$ the quantity null if two of the indexes are equal, and equal to the signature of $(\alpha_1, ..\alpha_r)$ if not,

- the exterior algebra ΛF of a vector space F is the set of anti-symmetric tensors :

$\lambda = \sum_{(\alpha_1,..\alpha_r)} \lambda_{\alpha_1...\alpha_r} e^{\alpha_1} \otimes ... \otimes e^{\alpha_r}$ with $\lambda_{\alpha_1...\alpha_r} = \epsilon(\alpha_1,..\alpha_r) \lambda_{s(\alpha_1...\alpha_r)}$ where (e^i) is a basis of F

$\Lambda F = \oplus_{r=0}^{n} \Lambda_r F$ notice that the field of scalars belong to ΛF

- ΛF becomes an algebra with the exterior product defined as :

$u, v \in F : u \wedge v = u \otimes v - v \otimes u$

$u_k \in F : u_1 \wedge u_2 ... \wedge u_r = \sum_{(\alpha_1,..\alpha_r)} \epsilon(\alpha_1,..\alpha_r) u_{\alpha_1} \otimes u_{\alpha_2} ... \otimes u_{\alpha_r}$

$(u^1 \wedge ... \wedge u^p) \wedge (u^{p+1} \wedge ... \wedge u^{p+q})$

$= \sum_{(\alpha_1,..\alpha_{p+q})} \epsilon(\alpha_1,..\alpha_{p+q}) u_{\alpha_1} \otimes u_{\alpha_2} ... \otimes u_{\alpha_{p+q}} = u^1 \wedge ... \wedge u^p \wedge u^{p+1} \wedge ... \wedge u^{p+q}$

In the antisymmetrization process I *do not use* the factor 1/r!.

- if $(e_i)_{i=1}^n$ is a basis of F, the set $e^{\alpha_1} \wedge ... \wedge e^{\alpha_r}$ of ordered products is a basis for $\Lambda_r F$ and an antisymmetric tensor :

$\lambda = \sum_{(\alpha_1,..\alpha_r)} \lambda_{\alpha_1...\alpha_r} e^{\alpha_1} \otimes ... \otimes e^{\alpha_r}$

$= \sum_{\{\alpha_1,..\alpha_r\}} \lambda_{\{\alpha_1...\alpha_r\}} \left(\sum_s \epsilon(\alpha_1,..\alpha_r) e^{\alpha_1} \otimes ... \otimes e^{\alpha_r} \right)$

$= \sum_{\{\alpha_1,..\alpha_r\}} \lambda_{\{\alpha_1...\alpha_r\}} e^{\alpha_1} \wedge ... \wedge e^{\alpha_r}$

and we have to pay heed to :

$\sum_{(\alpha_1,..\alpha_r)} \lambda_{\alpha_1...\alpha_r} e^{\alpha_1} \wedge ... \wedge e^{\alpha_r} = (r!) \sum_{\{\alpha_1,..\alpha_r\}} \lambda_{\{\alpha_1...\alpha_r\}} e^{\alpha_1} \wedge ... \wedge e^{\alpha_r}$

- with these conventions the exterior product of the p antisymmetric tensor λ_p and the q antisymmetric tensor μ_q is :

$\lambda_p \wedge \mu_q = \sum \lambda_{\{\alpha_1...\alpha_p\}} \mu_{\{\beta_1...\beta_q\}} e^{\alpha_1} \wedge ... \wedge e^{\alpha_p} \wedge e^{\beta_1} \wedge ... \wedge e^{\beta_q}$

This product is associative and $\lambda_p \wedge \mu_q = (-1)^{pq} \mu_q \wedge \lambda_p$

b) I will denote :

ϖ_0 the 4-form derived from a holonomic chart : $\varpi_0 = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$

so $\varpi_0 = \sum_{\alpha_1\alpha_2\alpha_3\alpha_4} \epsilon(\alpha_1, \alpha_2, \alpha_3, \alpha_4) dx^{\alpha_1} \otimes dx^{\alpha_2} \otimes dx^{\alpha_3} \otimes dx^{\alpha_4}$

The volume form deduced from a metric g is the following :

$$\varpi_4 = (\det O') \mathbf{dx}^0 \wedge \mathbf{dx}^1 \wedge \mathbf{dx}^2 \wedge \mathbf{dx}^3 = \partial^1 \wedge \partial^2 \wedge \partial^3 \wedge \partial^4 \quad (7)$$

The volume form on a manifold endowed with a metric g is defined as the 4-form such that an orthonormal basis has volume 1:

$\varpi_4 = \varpi_{4;0123} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$

$\varpi_4 (\partial_1, \partial_2, \partial_3, \partial_4)$

$= \varpi_{4;0123} (dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3) (\partial_1, \partial_2, \partial_3, \partial_4)$

$= \sum_{\alpha_1\alpha_2\alpha_3\alpha_4} \epsilon(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \varpi_{4;0123} (dx^{\alpha_1} \otimes dx^{\alpha_2} \otimes dx^{\alpha_3} \otimes dx^{\alpha_4}) (\partial_1, \partial_2, \partial_3, \partial_4)$

$= \varpi_{4;0123} \sum_{\alpha_1\alpha_2\alpha_3\alpha_4} \epsilon(\alpha_1, \alpha_2, \alpha_3, \alpha_4) O_1^{\alpha_1} O_2^{\alpha_2} O_3^{\alpha_3} O_4^{\alpha_4} = \varpi_{4;0123} \det O = 1$

$\Rightarrow \varpi_4 = (\det O')\, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$

Indeed we know that the volume form is $\varpi_4 = \sqrt{|\det g|} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$ and we have :

$[\eta] = [O]^t [g] [O] \Rightarrow (\det [O])^2 \det [g] = -1$

$\Rightarrow \det [g] = -\frac{1}{(\det[O])^2} = -(\det O')^2 \Rightarrow \sqrt{|\det g|} = |\det O'|$

The orthonormal bases are direct, so $|\det O'| = \det O'$

The volume form can be expressed in the orthonormal basis :

$\varpi_4 = \widehat{\varpi}_4 \partial^0 \wedge \partial^1 \wedge \partial^2 \wedge \partial^3 = \sum_{i_1 i_2 i_3 i_4} \epsilon(i_1, i_2, i_3, i_4)\, \widehat{\varpi}_4 \partial^{i_1} \otimes \partial^{i_2} \otimes \partial^{i_3} \otimes \partial^{i_4}$

and

$\varpi_4 (\partial_0, \partial_1, \partial_2, \partial_3) = \widehat{\varpi}_4 \sum_{i_1 i_2 i_3 i_4} \epsilon(i_1, i_2, i_3, i_4)\, \partial^{i_1} \otimes \partial^{i_2} \otimes \partial^{i_3} \otimes \partial^{i_4} (\partial_0, \partial_1, \partial_2, \partial_3)$

$= \widehat{\varpi}_4 \sum_{i_1 i_2 i_3 i_4} \epsilon(i_1, i_2, i_3, i_4)\, \delta_0^{i_1} \delta_1^{i_2} \delta_2^{i_3} \delta_3^{i_4} = \widehat{\varpi}_4 \det I_4 = 1 \blacksquare$

c) The partial derivative of the determinant is computed as follows :

$\partial_\alpha \sqrt{|\det g|} = \partial_\alpha \det O' = \sum_{\lambda,\mu} \left(\frac{d \det[O']}{d[O']^\lambda_\mu} \right) \partial_\alpha [O']^\lambda_\mu$

For any invertible matrix M one has : $\frac{d \det M}{d M^i_j} = [M^{-1}]^j_i \det M$ (the order of the indexes matters)

$\frac{d \det[O']}{d[O']^\lambda_\mu} = [O]^\mu_\lambda (\det [O'])$

$\partial_\alpha \sqrt{|\det g|} = \partial_\alpha \det [O'] = \sum_{\lambda,\mu} [O]^\mu_\lambda (\det [O'])\, \partial_\alpha [O']^\lambda_\mu = (\det [O'])\, Tr([O][\partial_\alpha O'])$

$$\partial_\alpha \sqrt{|\det g|} = (\det [O'])\, Tr([O][\partial_\alpha O']) \tag{8}$$

Notice also the identity : $\frac{\partial}{\partial O'^i_\alpha} = \frac{\partial O^\lambda_j}{\partial O'^i_\alpha} \frac{\partial}{\partial O^\lambda_j} = -O^\lambda_i O^\alpha_j \frac{\partial}{\partial O^\lambda_j}$

3.3 The other force fields

We shall be brief as we do not try to specify the force fields considered. The action of the force fields (other than gravitation) is represented through a principal connection **A** over U_M , its connection form denoted $\widehat{A} : U_M \to \Lambda_1 (TM^*; T_1^c U)$ and its potential $\grave{A} = \widehat{A} (\varphi_U (m, 1_U))$. Notice that the connection is valued in the complexified of the Lie algebra.

3.4 The fiber bundle of force fields

1) Both **G** and **A** can be defined as principal connection over Q_M. As equivariant connections they are essentially defined through their form, which

transforms in a gauge transformation as :

$q(m) = \varphi_Q(m,(1,1)) \to \widetilde{q} = \varphi_Q\left(m,(s,u)^{-1}\right)$

$G = \widehat{G}(\varphi_Q(m,(1,1))) = \sum_{a,\alpha} G^a_\alpha dx^\alpha \otimes \overrightarrow{\kappa}_a$

$\to \widetilde{G} = \widetilde{\widehat{G}}\left(\varphi_Q\left(m,(s,u)^{-1}\right)\right) = \sum_\alpha \left(Ad_{\mu(s)} \sum_a \left(\widehat{G}^\alpha_\alpha - \kappa^a_\alpha\right)\overrightarrow{\kappa}_a\right)\otimes dx^\alpha$

$\grave{A} = \widehat{\grave{A}}(\varphi_Q(m,(1,1))) = \sum_{a,\alpha} \grave{A}^a_\alpha dx^\alpha \otimes \overrightarrow{\theta}_a$

$\to \widetilde{\grave{A}} = \widetilde{\widehat{\grave{A}}}\left(\varphi_Q\left(m,(s,u)^{-1}\right)\right) = \sum_\alpha \left(Ad_u \sum_a \left(\widehat{\grave{A}}^\alpha_\alpha - \theta^a_\alpha\right)\overrightarrow{\theta}_a\right)\otimes dx^\alpha$

where

$\overrightarrow{\kappa}_\alpha = \kappa^a_\alpha \overrightarrow{\kappa}_a \in o(3,1) : \frac{d\mu(s)}{dx} = L'_{\mu(s)}(1)\kappa^a_\alpha dx^\alpha \otimes \overrightarrow{\kappa}_a$

$\overrightarrow{\theta}_\alpha = \theta^a_\alpha \overrightarrow{\theta}_a \in T_1^c G : \frac{du}{dx} = L'_u(1)\theta^a_\alpha dx^\alpha \otimes \overrightarrow{\theta}_a$

$\left(\widehat{\grave{A}},\widehat{G}\right) \to \left(\widetilde{\widehat{\grave{A}},\widehat{G}}\right) = \left(Ad_{\mu(s)}\left(\widehat{G}_\alpha - \overrightarrow{\kappa}_\alpha\right)dx^\alpha, Ad_u\left(\widehat{\grave{A}}_\alpha - \theta_\alpha\right)dx^\alpha\right)$

Similar to the vector bundle describing the states of particles, there is a fiber bundle describing the fields. But the relations above are affine and not simply linear, so this fiber bundle is an affine bundle and not a vector bundle.

2) From the linear representations $(o(3,1), Ad), (T_1U, Ad)$ of the groups Spin(3,1),U over their own algebra one builds the associated vector bundle : $F_M = Q_M \times_{Spin(3,1)\times U} (o(3,1) \times T_1U)$:

$\varphi_F(m,(1_{Spin},1_U)) = \left(\overrightarrow{\kappa}(m), \overrightarrow{\theta}(m)\right)$

$\sim \varphi_F\left(m,(s,u)^{-1}\right) = \left(Ad_s\overrightarrow{\kappa}(m), Ad_u\overrightarrow{\theta}(m)\right)$

A local basis of this vector bundle is given by a couple of vectors $\left(\overrightarrow{\kappa}_a(m), \overrightarrow{\theta}_b(m)\right)$

This real vector bundle can be extended to a half-complexified vector bundle to accomodate T_1^cU.

3) The 1-jet extension J^1F_M of this vector bundle is a fiber bundle coordinated by : $\left(x^\alpha, \left(\kappa^a, \theta^b\right), \left(\kappa^a_\alpha, \theta^b_\beta\right)\right)$. It is a vector bundle if restricted to the first two coordinates, meaning the bundle $J^1F_M \to M$, and an affine bundle with the last coordinate $J^1F_M \to F_M$.

Sections of the latter bundle can be seen as 1-form over M valued in F_M :

$\left(x^\alpha, \left(\kappa^a_\alpha(m), \theta^b_\beta(m)\right)\right) \leftrightarrow \left(\kappa^a_\alpha(m)\, dx^\alpha \otimes \overrightarrow{\kappa}_a(m), \theta^b_\beta(m)\, dx^\alpha \otimes \overrightarrow{\theta}_b(m)\right)$

Force fields are described by connections and defined by their potential which are 1-forms over M valued in the Lie algebras. There is a one-one cor-

respondance between principal connections over Q_M and equivariant sections over the affine bundle $J^1F_M \to F_M$ (Kolàr [14] IV.17).

3.5 The covariant derivative over E_M

1) The principal connections $(\mathbf{G},\mathbf{A})$ over Q_M induces a covariant derivative denoted ∇ acting on sections of the associated vector bundle E_M :

$\nabla\psi = \nabla\left(q,\psi^{ij}e_i\otimes f_j\right)$

$\nabla\psi = \left(\partial_\alpha\psi^{ij} + \vartheta'(1_{Spin(3,1)},1_U)\left(G_\alpha,\grave{A}_\alpha\right)\psi\right)dx^\alpha\otimes e_i\left(m\right)\otimes f_j\left(m\right)$

where :

$\vartheta'(1_{Spin(3,1)},1_U)\left(G_\alpha,\grave{A}_\alpha\right)\psi$

$= \sum_{j=1}^4\sum_{j=1}^m\left(\sum_{k=1}^4\sum_{a=1}^6 G_\alpha^a\left[\partial_a\rho\circ\Upsilon(1)\right]_k^i\psi^{kj} + \sum_{k=1}^m\sum_b \grave{A}_\alpha^b\left[\partial_b\chi\left(1\right)\right]_k^j\psi^{ik}\right)$

the index b running over the dimension of U.

$\nabla\psi = \sum\left(\partial_\alpha\psi^{ij} + G_\alpha^a\left[\partial_a\rho\circ\Upsilon'(1)\right]_k^i\psi^{kj} + \grave{A}_\alpha^a\left[\partial_a\chi\left(1\right)\right]_k^j\psi^{ik}\right)dx^\alpha\otimes e_i\otimes f_j$

It has the following properties :

$\forall\psi\in\Lambda_0\left(E_M\right),\forall X,Y\in TM,\forall\lambda,\mu\in\mathbb{R},f\in C(M;\mathbb{R})$:

$\nabla_{\lambda X+\mu Y}\psi = \lambda\nabla_X\psi + \mu\nabla_X\psi;\nabla_{fX}\psi = f\nabla_X\psi;$

$\nabla\left(f\psi\right) = f\left(\nabla\psi\right) + \left(d_Mf\right)\otimes\psi$

In a gauge transformations we get :

$\psi = \left(q,\psi^{ij}e_i\otimes f_j\right)\simeq\left(q\left(s,u\right)^{-1},\widetilde{\psi}^{kl}e_k\otimes f_l\right)$

$\psi^{ij}\to\widetilde{\psi}^{ij} = \left[\rho\left(s\right)\right]_l^i\left[\chi\left(g\right)\right]_m^j\psi^{lm}$

$G_\alpha\to\widetilde{G}_\alpha = Ad_s\left(B_\alpha-\zeta_\alpha\right)$

$\grave{A}_\alpha\to\widetilde{\grave{A}}_\alpha = Ad_g\left(\grave{A}_\alpha-\xi_\alpha\right)$

$\left(q\left(s,u\right)^{-1},\widetilde{\psi}^{kl}e_k\otimes f_l\right)$

$\simeq\left(q,\left(\partial_\alpha\psi^{ij} + \left[\rho\circ\Upsilon'(1)G_\alpha\right]_k^i\psi^{kj} + \left[\chi'\left(1\right)\grave{A}_\alpha\right]_k^j\psi^{ik}\right)dx^\alpha\otimes e_i\otimes f_j\right)$

Notation 3 *:*

ψ is a tensor, that will be conveniently represented as a matrix 4xm (it is not square) : $[\psi] = [\psi^{ij}]_{j=1..m}^{i=0..3}$

We have previously seen the square 4x4 matrices :$[\kappa_a] = (\rho\circ\Upsilon)'\left(1\right)\left(\widehat{\jmath}\left(\overrightarrow{\kappa}_a\right)\right)$. We will denote the square 4x4 matrices : $\sum_{a=1}^4\left[\kappa_a\right]G_\alpha^a = [G_\alpha]$. Notice that $\left[\widetilde{G}_\alpha\right] = \sum_a\left[\widetilde{\kappa}_a\right]G_\alpha^a\in o(3,1)$

$\chi'(1) : T_1^c U \to L_C(W;W)$ is a complex linear map. $\partial_a \chi(1_G) \in L_C(W;W)$ is represented with the (f_k) basis by a square mxm matrix with complex coefficients. We will denote the square mxm matrices : $\partial_a \chi(1_G) = \chi'(1_G) \overrightarrow{\theta}_a = [\theta_a]$.and $\sum_a [\theta_a] \grave{A}_\alpha^a = \left[\grave{A}_\alpha\right]$. The (W,χ) representation being unitary : $\chi'(1) = -\chi'(1)^*$ and $[\theta_a] = -[\theta_a]^*$.

$$\nabla\psi = \sum_{\alpha ijk} \left(\partial_\alpha \psi^{ij} + [\kappa_a]_k^i G_\alpha^a \psi^{kj} + [\theta_a]_k^j \grave{A}_\alpha^a \psi^{ik}\right) dx^\alpha \otimes e_i(m) \otimes f_j(m) \quad (9)$$

So the covariant derivative reads in matrix notation :

$$[\nabla_\alpha \psi] = [\partial_\alpha \psi] + [G_\alpha][\psi] + [\psi]\left[\grave{A}_\alpha\right]^t \quad (10)$$

3) The covariant derivative gives a parallel transport over E_M along a path m(t) in M with the condition : $\nabla_{m'(t)}\psi(m(t)) = 0$. Practically the observer must stay in a path such that the effects of external fields do not change. Or equivalently two different observers proceeding to the same experiment in similar conditions with a test particle shall get equivariant measures. So, in principle, there is a way for these two obervers to calibrate their instruments, that is to know where their basis $e_i(m) \otimes f_j(m)$ stand relatively to each other.

4) The covariant derivative acts on the section ψ, the kinematic and the physical characteristics. But as a 1-form on M it acts on the velocity, which a vector in $T_q M$. All these actions are local and linear, as expected. We will have a better look at the mechanisms involved in the 4th part, until then we will stay at a general level.

3.6 Interaction Field/Field

The force fields interact with each other. At this step we will not enter into a precise description of the mechanisms involved, but just introduce one key ingredient : the curvature. In the principle of least action picture we need derivatives of the various quantities. For the states of particles which are sections of associated bundle that is the covariant derivative. The force fields are described as potential (G,À), which are 1-form over M, so we need some kind of covariant derivative for forms.

3.6.1 Exterior covariant derivative on principal bundles

The force fields are fully described in the principal bundle picture, so only these bundles are involved here.

1) The bracket of forms on M valued in a Lie algebra is defined as follows :

$\lambda \in \Lambda_p(M;T_1U), \mu \in \Lambda_q(M;T_1^cU)$

$\rightarrow [\lambda,\mu] = \sum_{a,b,c} C^a_{bc}\lambda^b \wedge \mu^c \otimes \overrightarrow{\theta}_a \in \Lambda_{p+q}(M;T_1U^c)$

$\lambda \in \Lambda_p(M;o(3,1)), \mu \in \Lambda_q(M;o(3,1))$

$\rightarrow [\lambda,\mu] = \sum_{a,b,c} G^a_{bc}\lambda^b \wedge \mu^c \otimes \overrightarrow{\kappa}_a \in \Lambda_{p+q}(M;o(3,1))$

where C^a_{bc}, G^a_{bc} are the structure coefficients of the algebras (they are real numbers in both cases).

The exterior covariant derivative of a p-form on M valued in the Lie algebra is defined as :

$\lambda \in \Lambda_p(M;T_1^cU) : \nabla_e\lambda = d_M\lambda + \left[\grave{A},\lambda\right]$

$\lambda \in \Lambda_p(M;o(3,1)) : \nabla_e\lambda = d_M\lambda + [G,\lambda]$

where $d_M\lambda$ is the usual exterior derivative of the p-form on M.

2) The potential is a 1-form, so one can compute its exterior covariant derivative :

$\mathcal{F}_A = \nabla_e\grave{A} = d_V\left(\grave{A}\right) + \left[\grave{A},\grave{A}\right]_{T_1U} \in \Lambda_2(M;T_1^cU)$

$\mathcal{F}_G = \nabla_e G = d_V(G) + [G,G]_{o(3,1)} \in \Lambda_2(M;o(3,1))$

They are 2-forms valued in the Lie algebra, expressed in components as :

$\mathcal{F}_A = \sum_{\{\alpha,\beta\},a} \mathcal{F}^a_{A\{\alpha\beta\}} dx^\alpha \wedge dx^\beta \otimes \overrightarrow{\theta}_a$

with $\mathcal{F}^a_{A\alpha\beta} = \partial_\alpha\grave{A}^a_\beta - \partial_\beta\grave{A}^a_\alpha + \sum_{bc} C^a_{bc}\grave{A}^b_\alpha\grave{A}^c_\beta$

$\mathcal{F}_G = \sum_{\{\alpha,\beta\},a} \mathcal{F}^a_{G\{\alpha\beta\}} dx^\alpha \wedge dx^\beta \otimes \overrightarrow{\kappa}_a$

with $\mathcal{F}^a_{G\alpha\beta} = \partial_\alpha G^a_\beta - \partial_\beta G^a_\alpha + \sum_{bc} G^a_{bc}G^b_\alpha G^c_\beta$

and the usual notation $\{\alpha,\beta\}$ for an ordered set of indexes.

Their exterior covariant derivative is null : $\nabla_e\mathcal{F}_A = 0; \nabla_e\mathcal{F}_G = 0$

In a gauge transformation these forms transform as :

$\mathcal{F}_A\left(q(m)(s,u)^{-1}\right) = Ad_u\mathcal{F}_A(q(m))$

$\mathcal{F}_G\left(q(m)(s,u)^{-1}\right) = Ad_s\mathcal{F}_G(q(m))$

The definitions and names for these quantities vary in the litterature. We will call them, in these definitions, the curvature forms.

3) They are the quantities (and possibly their derivatives) which should be put in the lagrangian to account for the interactions between force fields. In General Relativity it is usual to use the Riemann tensor and the scalar curvature at this effect, so it is useful to see how these quantities are related to our curvature forms. As previously with affine connections and principal connexions the link goes through the associated vector bundle.

3.6.2 Covariant exterior derivative on associated vector bundle

1) For any covariant derivative ∇ on a vector bundle G_M [2] there is a unique extension as a linear operator

$\nabla_e : \Lambda_r(M;G_M) \to \Lambda_{r+1}(M;G_M)$ on the forms valued in G_M , such that :

$\forall \lambda \in \Lambda_r(M;\mathbb{R}), \pi \in \Lambda_r(M;G_M) : \nabla_e (\lambda \wedge \pi) = (d_M \lambda) \otimes \pi + (-1)^r \lambda \wedge \nabla_e \pi$

It is defined (Husemoller [10] 19.2) by :

$\pi \in \Lambda_r(M;G_M) : \pi = \sum_i \sum_{\{\alpha_0 ... \alpha_{r-1}\}} \pi^i_{\{\alpha_0 ... \alpha_{r-1}\}} dx^{\alpha_0} \wedge .. \wedge dx^{\alpha_{r-1}} \otimes \partial_i (m)$

$\nabla_e \pi = \sum_{i,\alpha} \sum_{\{\alpha_0 ... \alpha_{r-1}\}} \left(\partial_\alpha \pi^i_{\{\alpha_0 ... \alpha_{r-1}\}} + \left[\widetilde{G}_\alpha \right]^i_k \pi^k_{\{\alpha_0 ... \alpha_{r-1}\}} \right) dx^\alpha \wedge dx^{\alpha_0} \wedge ..dx^{\alpha_{r-1}} \otimes \partial_i (m)$

$\Leftrightarrow \nabla_e \pi = \sum_i \left(d_M \pi^i + \sum_k \left(\sum_\alpha \left[\widetilde{G}_\alpha \right]^i_k dx^\alpha \right) \wedge \pi^k \right) \otimes \partial_i$ where d_M is the usual exterior differential on M.

2) If one applies two times this operator on the same form :

$\nabla^2_e \pi = \left(\sum_k [\digamma]^i_k \wedge \pi^k \right) \otimes \partial_i$

where $\digamma = \jmath'(1)(\mathcal{F}_G) = \sum_{\alpha<\beta} \sum_a \mathcal{F}^a_{G\{\alpha\beta\}} [\widetilde{\kappa}_a] dx^\alpha \wedge dx^\beta$ is a 2-form on M valued in the linear maps over G_M and represented in the canonic basis of $\mathbb{R}^4$ by matrices of o(3,1). $\digamma$ is nothing other than the curvature form $\mathcal{F}_G$ expressed in the orthonormal basis :

$\digamma = \frac{1}{2} \sum_{\alpha\beta} \sum_a \mathcal{F}^a_{G\alpha\beta} [\widetilde{\kappa}_a] dx^\alpha \wedge dx^\beta = \sum_{\{ij\}kl} \digamma^l_{\{ij\}k} \partial^i \wedge \partial^j \otimes \partial^k \otimes \partial_l$

with $\digamma^l_{ijk} = \sum_{a\alpha\beta} \mathcal{F}^a_{G\alpha\beta} [\widetilde{\kappa}_a]^l_k O^\alpha_i O^\beta_j \Leftrightarrow \sum_{a\alpha\beta} \mathcal{F}^a_{G\alpha\beta} [\widetilde{\kappa}_a]^l_k = \sum \digamma^l_{ijk} O'^i_\alpha O'^j_\beta$

It can be shown that $\nabla_e \digamma_i = 0$ where

$\digamma_i = \sum_{a\{\alpha\beta\}} \mathcal{F}^a_{G\alpha\beta} [\widetilde{\kappa}_a]^j_i dx^\alpha \wedge dx^\beta \otimes \partial_j$

[2] We take here the G_M associated vector bundle but the procedure is general

3) The same calculation can be done with any covariant derivative on a vector bundle. With an affine connection on TM defined by the Christofell coefficients $\Gamma^{\beta}_{\alpha\gamma}$ one gets :

$\nabla_e \left(\sum_{\alpha,\{\alpha_1...\alpha_r\}} \pi^{\alpha}_{\{\alpha_1...\alpha_r\}} \left(dx^{\alpha_1} \wedge ... \wedge dx^{\alpha_r} \right) \otimes \partial_\alpha \right)$

$= \sum_\alpha \left(d\pi^\alpha + \sum_\beta \left(\Gamma^{\alpha}_{\gamma\beta} dx^\gamma \right) \wedge \pi^\beta \right) \otimes \partial_\alpha$

$\nabla^2_e \left(\sum_{\alpha,\{\alpha_1...\alpha_r\}} \pi^{\alpha}_{\{\alpha_1...\alpha_r\}} \left(dx^{\alpha_1} \wedge ... \wedge dx^{\alpha_r} \right) \otimes \partial_\alpha \right)$

$= \sum_\alpha \sum_\beta \left(d \left(\Gamma^{\alpha}_{\gamma\beta} dx^\gamma \right) + \left(\sum_\gamma \left(\Gamma^{\alpha}_{\lambda\gamma} dx^\lambda \right) \wedge \left(\Gamma^{\gamma}_{\mu\beta} dx^\mu \right) \right) \right) \wedge \pi^\beta \otimes \partial_\alpha$

The quantity

$\sum_\alpha \sum_\beta \left(d \left(\Gamma^{\alpha}_{\gamma\beta} dx^\gamma \right) + \left(\sum_\gamma \left(\Gamma^{\alpha}_{\lambda\gamma} dx^\lambda \right) \wedge \left(\Gamma^{\gamma}_{\mu\beta} dx^\mu \right) \right) \right) \otimes \partial_\alpha$

$= \sum_{\alpha\beta} \sum_{\gamma<\eta} R^{\alpha}_{\{\gamma\eta\}\beta} dx^\gamma \wedge dx^\eta \otimes dx^\beta \otimes \partial_\alpha$ i

is the Riemann tensor : $R^{\alpha}_{\gamma\eta\beta} = \partial_\gamma \Gamma^{\alpha}_{\eta\beta} - \partial_\eta \Gamma^{\alpha}_{\gamma\beta} + \Gamma^{\alpha}_{\gamma\varepsilon} \Gamma^{\varepsilon}_{\eta\beta} - \Gamma^{\alpha}_{\eta\varepsilon} \Gamma^{\varepsilon}_{\gamma\beta}$ which can be seen as the curvature form of the affine connection and

$\nabla^2_e (\pi) = \sum_{\alpha\beta} \sum_{\gamma<\eta} R^{\alpha}_{\{\gamma\eta\}\beta} \wedge \pi^\beta \otimes \partial_\alpha$.

From the Riemann tensor one deduces the Ricci tensor :

$Ric_{\gamma\beta} dx^\gamma \otimes dx^\beta = dx^\alpha \left(R \left(\partial_\eta \right) \right) :: Ric_{\alpha\beta} = \sum_\gamma R^{\gamma}_{\alpha\gamma\beta}$

Notice that these calculations can be done without any reference to a metric.

Now with a metric g there is the scalar curvature : $R = \sum_{\alpha\beta} g^{\alpha\beta} Ric_{\alpha\beta}$

The Ricci tensor is symmetric if the connection is symmetric, but the scalar curvature has a unique definition :

$R = \frac{1}{2} \sum_{\alpha\beta} g^{\alpha\beta} \left(Ric_{\alpha\beta} + Ric_{\beta\alpha} \right) = \frac{1}{2} \left(\sum_{\alpha\beta} g^{\alpha\beta} Ric_{\alpha\beta} + g^{\alpha\beta} Ric_{\beta\alpha} \right)$

$= \frac{1}{2} \left(\sum_{\alpha\beta} g^{\alpha\beta} Ric_{\alpha\beta} + g^{\beta\alpha} Ric_{\beta\alpha} \right) = \sum_{\alpha\beta} g^{\alpha\beta} Ric_{\alpha\beta}$

4) We have seen previously that for a principal connection there is a unique affine connection with $[\Gamma_\gamma] = \left([O] \left[\widetilde{G}_\gamma \right] - \partial_\gamma [O] \right) [O']$ from which one can compute as above from the connection $\mathbf{G}$: :

the Riemann tensor : $[R_{\gamma\eta}]$

$= \partial_\gamma [\Gamma_\eta] - \partial_\eta [\Gamma_\gamma] + [[\Gamma_\gamma], [\Gamma_\eta]]$.

$= [O] \left(\left[\widetilde{\partial_\gamma G_\eta} \right] - \left[\widetilde{\partial_\eta G_\gamma} \right] + \left[\widetilde{G_\gamma} \right] \left[\widetilde{G_\eta} \right] - \left[\widetilde{G_\eta} \right] \left[\widetilde{G_\gamma} \right] \right) [O']$

$- [O] \left[\widetilde{G_\eta} \right] [O'] [\partial_\gamma O] [O'] + [O] \left[\widetilde{G_\eta} \right] [O'] [\partial_\gamma O] [O'] + [O] \left[\widetilde{G_\gamma} \right] [O'] [\partial_\eta O] [O']$

$- [O] \left[\widetilde{G_\gamma} \right] [O'] [\partial_\eta O] [O'] + [\partial_\eta O] [O'] [\partial_\gamma O] [O'] - [\partial_\eta O] [O'] [\partial_\gamma O] [O']$

$-[\partial_\gamma O][O'][\partial_\eta O][O'] + [\partial_\gamma O][O'][\partial_\eta O][O']$

$[R_{\gamma\eta}] = [O]\left[\widetilde{\mathcal{F}_{G\gamma\eta}}\right][O']$

So the Riemann tensor associated to the connection $\mathbf{G}$ is :

$R^\eta_{\alpha\beta\gamma} = \sum_{ij} O^\eta_i [\mathcal{F}_{G\alpha\beta}]^i_j O'^j_\gamma \Leftrightarrow [\mathcal{F}_{G\alpha\beta}]^i_j = R^\eta_{\alpha\beta\gamma} O'^i_\eta O^\gamma_j$

that is the tensor F, which is the curvature form expressed in the orthonormal basis.

the Ricci tensor : $Ric = \sum_{\alpha\beta} Ric_{\alpha\beta} dx^\alpha \otimes dx^\beta = \sum_{ij\gamma} O^\gamma_i \left[\widetilde{\mathcal{F}_{G\alpha\gamma}}\right]^i_j O'^j_\beta dx^\alpha \otimes dx^\beta$

$Ric_{\alpha\beta} = \sum_{ij\gamma} \left[\widetilde{\mathcal{F}_{G\alpha\gamma}}\right]^i_j O^\gamma_i O'^j_\beta = \sum_{aij\gamma} [\widetilde{\kappa}_a]^i_j O^\gamma_i O'^j_\beta \mathcal{F}^a_{G\alpha\gamma}$

$= \sum_{ij\gamma} (\delta^{ip_a}\delta_{jq_a} - \delta^{iq_a}\eta_{jp_a}) O^\gamma_i O'^j_\beta \mathcal{F}^a_{G\alpha\gamma}$

$= \sum_{ij\gamma} \left(O^\gamma_{p_a} O'^{q_a}_\beta - \eta_{p_a p_a} O^\gamma_{q_a} O'^{p_a}_\beta\right) \mathcal{F}^a_{G\alpha\gamma}$

In the orthonormal frame the Ricci tensor from the connection $\mathbf{G}$: $Ric_{ij} = \sum_{k\alpha\gamma} [\mathcal{F}_{G\alpha\gamma}]^k_j O^\alpha_i O^\gamma_k = \sum_k F^k_{ikj}$

the scalar curvature : With the metric defined as : $g^{\alpha\beta} = \eta^{kl} O^\alpha_k O^\beta_l$:

$R = \sum \eta^{kl} O^\alpha_k O^\beta_l \left[\widetilde{\mathcal{F}_{G\alpha\gamma}}\right]^i_j O^\gamma_i O'^j_\beta = \sum \eta^{kj} \left[\widetilde{\mathcal{F}_{G\alpha\beta}}\right]^i_j O^\beta_i O^\alpha_k$

$= \sum_{a\alpha\beta ij} \mathcal{F}^a_{G\alpha\beta} ([\widetilde{\kappa}_a][\eta])^i_j O^\beta_i O^\alpha_j = \sum_{\alpha\beta ij} \mathcal{F}^a_{G\alpha\beta} (\delta^{ip_a}\delta_{jq_a} - \delta^{iq_a}\delta_{jp_a}) O^\beta_i O^\alpha_j$

$= \sum_{a\alpha\beta} \mathcal{F}^a_{G\alpha\beta} \left(O^\beta_{p_a} O^\alpha_{q_a} - O^\beta_{q_a} O^\alpha_{p_a}\right) = 2\sum_{a,\alpha<\beta} \mathcal{F}^a_{G\alpha\beta} \left(O^\beta_{p_a} O^\alpha_{q_a} - O^\beta_{q_a} O^\alpha_{p_a}\right)$

$$\mathbf{R} = \sum_{a\alpha\beta} \mathcal{F}^a_{G\alpha\beta} \left(O^\beta_{p_a} O^\alpha_{q_a} - O^\beta_{q_a} O^\alpha_{p_a}\right) = \mathbf{2} \sum_{a,\alpha<\beta} \mathcal{F}^a_{G\alpha\beta} \left(O^\beta_{p_a} O^\alpha_{q_a} - O^\beta_{q_a} O^\alpha_{p_a}\right) \quad (11)$$

In the orthonormal frame :

$R = \frac{1}{2} \sum_{akl\alpha\beta} \left(F^q_{pqj}\eta^{pj} + F^q_{pqj}\eta^{jp}\right) = \sum_{pqj} ([F_{pq}][\eta])^q_p = \sum_{ijk} [F(\partial_p, \partial_q)]^q_j \eta^{jp}$

$= \sum_i [F(\partial_1, \partial_i)]^i_1 + [F(\partial_2, \partial_i)]^i_2 + [F(\partial_3, \partial_i)]^i_3 - [F(\partial_0, \partial_i)]^i_0$

If the connection $\mathbf{G}$ is symmetric this scalar curvature will be identical to the usual quantity computed from g.

This quantity is preserved by a gauge transformation.

$[O] \to [O][\mu(s)]^{-1}$

$\left[\widetilde{\mathcal{F}_{G\alpha\gamma}}\right] \to Ad_{\mu(s)} \left[\widetilde{\mathcal{F}_{B\alpha\gamma}}\right] = [\mu(s)] \left[\widetilde{\mathcal{F}_{G\alpha\gamma}}\right] [\mu(s)]^{-1}$

$\sum [O]^\gamma_i \left[\widetilde{\mathcal{F}_{G\alpha\gamma}}\right]^i_j \eta^{kj} [O]^\alpha_k$

$\rightarrow \sum [O]_a^\gamma \left[\mu(s)^{-1}\right]_i^a [\mu(s)]_b^i \left[\widetilde{\mathcal{F}_{G\alpha\gamma}}\right]_c^b \left[\mu(s)^{-1}\right]_j^c [\eta]_j^k [O]_d^\alpha \left[\mu(s)^{-1}\right]_k^d$
$= \sum [O]_a^\gamma \left[\widetilde{\mathcal{F}_{G\alpha\gamma}}\right]_c^a [O]_d^\alpha \left[\mu(s)^{-1}\right]_k^d [\eta]_j^k \left[\left(\mu(s)^{-1}\right)^t\right]_c^j$
$= \sum [O]_a^\gamma \left[\widetilde{\mathcal{F}_{G\alpha\gamma}}\right]_c^a [O]_d^\alpha \eta^{dc}$
where we use the property of $\left[\mu(s)^{-1}\right] \in SO(3,1)$■

3.6.3 Torsion

While we are in these calculations I take the opportunity to introduce the torsion tensor and the structure coefficients which will be useful later on.

1) The fundamental form is the 1-form on M valued in G_M :

$$\Theta = \sum_{i=0}^{3} \partial^i \otimes \partial_i = \sum_{i\alpha} O'^i_\alpha dx^\alpha \otimes \partial_i \in \Lambda_1(M; G_M) \tag{12}$$

This is a purely geometric quantity, independant from any connection.

2) The components of its exterior derivative $d_M\Theta$ are the structure coefficients $c^i_{lk} = [\partial_l, \partial_k]^i$ of the algebra of vectors in the basis ∂_i (the brackets are the commutators of the vector fields $\partial_i = O_i^\alpha \partial_\alpha$) :

$$\mathbf{c}^i_{lk} = [\partial_l, \partial_k]^i = \sum_{\alpha\beta} \mathbf{O}'^i_\beta \left(O_l^\alpha \partial_\alpha O_k^\beta - O_k^\alpha \partial_\alpha O_l^\beta\right) \tag{13}$$

$d\Theta = \sum_r \sum_{\alpha,\beta} \left(\partial_\beta O'^r_\alpha\right) \partial^\beta \wedge \partial^\alpha \otimes \partial_r$
$= \sum_r \sum_{p,q} c^r_{pq} \partial^p \wedge \partial^q \otimes \partial_r = 2 \sum_r \sum_{p<q} c^r_{pq} \partial^p \wedge \partial^q \otimes \partial_r$

We have $c^r_{pq} = -c^r_{qp}$ so it is convenient to choose an order for the indexes. Using the correspondance between the indexes in the basis of o(3,1) and the couples (p,q) (see table 1) we will denote :

$$a = 1..6, r = 0,..3 : c^r_a = c^r_{p_a q_a} : \tag{14}$$

$$c^r_1 = c^r_{32}, c^r_2 = c^r_{13}, c^r_3 = c^r_{21}, c^r_4 = c^r_{01}, c^r_5 = c^r_{02}, c^r_6 = c^r_{03} \tag{15}$$

With this notation we have : $d\Theta = 2 \sum_{a=1}^{6} \sum_{r=0}^{3} c^r_a \partial^{p_a} \wedge \partial^{q_a} \otimes \partial_r$

3) The exterior covariant derivative of Θ is :

$\nabla_e \Theta = \sum_r \left(d_M \Theta^r + \sum_i \left(\left(\sum_\alpha \left[\widetilde{G}_\alpha \right]_i^r dx^\alpha \right) \wedge \Theta^i \right) \right) \otimes \partial_r$

$= \left(d_M \left(\sum_\alpha O_\alpha'^r \partial^\alpha \right) + \sum_i \left(\left(\sum_\alpha \left[\widetilde{G}_\alpha \right]_i^r dx^\alpha \right) \wedge \left(\sum_\beta O_\beta'^i \partial^\beta \right) \right) \right) \otimes \partial_r$

$= \sum_r \left(\sum_{\alpha\beta} \left(\partial_\alpha O_\beta'^r + \left[\widetilde{G}_\alpha \right]_i^r O_\beta'^i \right) dx^\alpha \wedge dx^\beta \otimes \partial_r \right)$

$= 2 \sum_{r\gamma} \sum_{\alpha<\beta} \left(\partial_\alpha O_\beta'^r - \partial_\beta O_\alpha'^r + \left[\widetilde{G}_\alpha \right]_i^r O_\beta'^i - \left[\widetilde{G}_\beta \right]_i^r O_\alpha'^i \right) O_r^\gamma dx^\alpha \wedge dx^\beta \otimes \partial_\gamma$

Expressed with $[\Gamma_\alpha] = \left([O] \left[\widetilde{G}_\alpha \right] - [\partial_\alpha O] \right) [O']$ it gives:

$\nabla_e \Theta = \sum_{\alpha<\beta} \left(\Gamma_{\alpha\beta}^\gamma - \Gamma_{\beta\alpha}^\gamma \right) dx^\alpha \wedge dx^\beta \otimes \partial_\gamma$

which is the usual torsion 2-form of the affine connection associated to **G.** This connection is symmetric iff $\nabla_e \Theta = 0$ and we have the relation : $\nabla_e^2 \Theta = \digamma \wedge \Theta$.

4) The torsion tensor can be expressed as :

$\nabla_e \Theta = d\Theta + \sum_{i\alpha\beta r} \left(\left[\widetilde{G}_\alpha \right]_i^r O_\beta'^i \right) dx^\alpha \wedge dx^\beta \otimes \partial_r$

with $d\Theta = 2 \sum_r \sum_{p<q} c_{pq}^r \partial^p \wedge \partial^q \otimes \partial_r = 2 \sum_{a=1}^6 \sum_{r=0}^3 c_a^r \partial^{p_a} \wedge \partial^{q_a} \otimes \partial_r$

and

$\sum_{i\alpha\beta r} \left(\left[\widetilde{G}_\alpha \right]_i^r O_\beta'^i \right) dx^\alpha \wedge dx^\beta \otimes \partial_r$

$= \sum_{\alpha\beta ijkr} \left[\widetilde{G}_\alpha \right]_i^r O_\beta'^i O_j^\alpha O_k^\beta \partial^j \wedge \partial^k \otimes \partial_r$

$= \sum_r \sum_{ij} \left[\widetilde{G}_j \right]_i^r \partial^j \wedge \partial^i \otimes \partial_r$

$= \sum_{a=1}^6 \sum_{r=0}^3 \left(\left[\widetilde{G}_{p_a} \right]_{q_a}^r - \left[\widetilde{G}_{q_a} \right]_{p_a}^r \right) \partial^{p_a} \wedge \partial^{q_a} \otimes \partial_r$

$= \sum_{a=1}^6 \sum_{r=0}^3 T^{ar} \partial^{p_a} \wedge \partial^{q_a} \otimes \partial_r$

with : $\left[\widetilde{G}_j \right] = \sum_\alpha \left[\widetilde{G}_\alpha \right] O_j^\alpha$.

The table T^{ar} indexed on a and r is computed with $[\widetilde{\kappa}_a]_j^i = \left(\delta^{ip_a} \delta_{jq_a} - \delta^{iq_a} \eta_{jp_a} \right)$:

TABLE 2: $T^{ar} =$

$$\begin{bmatrix} a \setminus r & p & q & 0 & 1 & 2 & 3 \\ 1 & 3 & 2 & -G_2^6 + G_3^5 & -G_2^2 - G_3^3 & G_2^1 & G_3^1 \\ 2 & 1 & 3 & G_1^6 - G_3^4 & G_1^2 & -G_1^1 - G_3^3 & G_3^2 \\ 3 & 2 & 1 & -G_1^5 + G_2^4 & G_1^3 & G_2^3 & -G_1^1 - G_2^2 \\ 4 & 4 & 1 & G_0^4 & -G_1^4 & -G_1^5 + G_0^3 & -G_0^2 - G_1^6 \\ 5 & 4 & 2 & G_0^5 & -G_2^4 - G_0^3 & -G_2^5 & G_0 - G_2^6 \\ 6 & 4 & 3 & G_0^6 & G_0^2 - G_3^4 & -G_0^1 - G_3^5 & -G_3^6 \end{bmatrix}$$

and we have :

$\nabla_e \Theta = \sum_{r,a} \left(2c_a^r + T^{ar}\right) \partial^{p_a} \wedge \partial^{q_a} \otimes \partial_r$

It should be noticed that the torsion, as the scalar curvature, are related to a connection : they are not some geometrical properties of the manifold M. They are computed composed either with $\left(O_{p_a}^{\beta} O_{q_a}^{\alpha} - O_{q_a}^{\beta} O_{p_a}^{\alpha}\right)$ or $O_{\beta}'^{r}\left(O_{p_a}^{\alpha} \partial_\alpha O_{q_a}^{\beta} - O_{q_a}^{\alpha} \partial_\alpha O_{p_a}^{\beta}\right)$ which are pure geometrical quantities. As a manifold there are topological obstructions to the existence of a structure of principal fiber bundle, but given a fiber bundle there is no unique compatible connection. Indeed according to General Relativity the connection is fixed through interactions with the content (matter and fields) of the universe. Assuming that there is some intrinsic scalar curvature or torsion of the universe would state that the vacuum has an pre-existing physical structure.

3.6.4 Remark

In the tetrad method it is usual to introduce the quantities called :

a) the 1-connection 1-form : $\Omega : \Omega_j^i = \left[\widetilde{G}_j\right]_k^i \partial^k \in \Lambda(M; L(n))$

b) the 2-form torsion : $T^i = \dfrac{1}{2} T_{jk}^i \partial^j \Lambda \partial^k = \sum_{\{jk\}} T_{jk}^i \partial^j \Lambda \partial^k$;

$T_{jk}^i = T(\partial_j, \partial_k)\partial^i = \left[\widetilde{G}_j\right]_k^i - \left[\widetilde{G}_k\right]_j^i - c_{jk}^i$

c) the 2-form curvature : $R_j^i = \dfrac{1}{2} R_{jkl}^i \partial^k \Lambda \partial^l = \sum_{\{kl\}} R_{jkl}^i \partial^k \Lambda \partial^l = \mathcal{F}$

And the following relations :

Cartan's structure equations :

$d\partial^i + \Omega_j^i \Lambda \partial^j = T^i \Leftrightarrow T = \nabla_e \Theta$

$d\Omega_j^i + \Omega_k^i \Lambda \Omega_j^k = R_j^i \Leftrightarrow \mathcal{F}_G = \nabla_e G$

Bianchi's identities :

$dT^a + \Omega_j^a \Lambda T^j = R_j^a \Lambda \partial^j \Rightarrow dT^a \partial_a + \Omega_j^a \partial_a \Lambda T^j \partial_j = \left(R_j^a \partial^j \otimes \partial_a\right) \Lambda \partial^j \Leftrightarrow \nabla_e T = \mathcal{F} \wedge \Theta = \nabla_e^2 \Theta$

$$dR^a_j + \Omega^a_c \Lambda R^c_j = R^a_c \Lambda \Omega^c_j \Leftrightarrow \nabla_e R = \nabla^2 G = F \wedge G$$

Part II
LAGRANGIAN

4 PRINCIPLES

4.1 The point particle issue

1) The system is described at each time t by the following quantities, measured on the hypersurface S(t) :

- the geometry: the matrix $[O(m)]$ or equivalently the fundamental form Θ

- the state of the N particles : $\psi_n(q_n(t)) \in E_M$ located a some point $q_n(t) \in S(t)$

- the potential of the gravitation G and the other fields À on S(t)

For the complex variables the complex conjugate should also be involved. The lagrangian is a real function, R-differentiable but not C-differentiable : if it was holomorphic the partial derivatives with the conjugate variables would be null. So we must consider separately the real and the imaginary part.

Let us denote all these variables z^j where j runs over all the variables and their coordinates. The lagrangian should also include their partial derivatives : $\frac{\partial z^i}{\partial x^\alpha}$. As the interactions are described by first order connections it is sensible to limit these partial derivatives to the first order also.

A general field model shall cover both the "vacuum" - no particles - and the "free particles" - no force fields - cases. So the action can be split in one part denoted S_M addressing particles and interacting fields, and another part S_F addressing the interacting fields only. ψ and its derivatives figure in the S_M part only.

2) The concept of point particle raises many difficulties in field theories. They arise for the determination of the Lagrange equations (Poisson [21]) and the trajectories (Quinn [22]). There are some ways to circumvent these problems, but they are rather cumbersome and involve methods with which one cannot be fully comfortable (such as fields propagation coming from the future). In fact these issues appear in the case of a single particle interacting with its own field, which is a simple model, but perhaps an unphysical one.

Without pretending to settle this issue I assume that the force fields are well defined sections of the bundle J^1F_M and the action S_F reads :

$S_F = \int_\Omega L_F \left(G^a_\alpha, \partial_\beta G^a_\alpha, \operatorname{Re} \grave{A}^a_\alpha, \operatorname{Re} \partial_\beta \grave{A}^a_\alpha, \operatorname{Im} \grave{A}^a_\alpha, \operatorname{Im} \partial_\beta \grave{A}^a_\alpha z^i (m), O'^i_\alpha, \partial_\beta O'^i_\alpha \right) \varpi_4$

with some real valued function L_F. Notice this is ϖ_4 in the integral ($\varpi_4 = \sqrt{|\det g|}\varpi_0$)

3) The action S_M should be some integral like :

$S_M = \sum_{k=1}^{N_t}$

$\int_0^T L_M \left(v^\alpha_k (t), \operatorname{Re} \psi^{ij}_k (t), \operatorname{Im} \psi^{ij}_k (t), \operatorname{Re} \partial_\alpha \psi^{ij}_t (t), \operatorname{Im} \partial_\alpha \psi^{ij}_k (t), \operatorname{Re} \grave{A}^a_\alpha (q_k(t)), ... \right) dt$

with the sum over the N_t particles present at t, their trajectories $q_k(t)$ within Ω and their velocities : $v_k(t) = \frac{dq_k}{dt}$.

It is assumed that Ω is large enough so that any particle entering into the system stays within, or changes into another one (which is just a change of the value of ψ) or is annihiliated : there is no particle leaving or entering Ω during the whole period of observation [0,T] except at t=0 or t=T. Conversely particles can be created "from the vacuum".

4) Let us consider first the case where all the particles live over [0,T]. If the problem has a "well posed" initial value formulation, that we will assume, the principle of least action leads to equations such that the trajectories $q_k(t)$ are uniquely determined from the initial values, notably the positions a_k of the particle k at t=0. Implementing a classical method attributed to Low, one can therefore assume that there is some function $\widehat{f} : S(0) \times [0,T] \to \Omega$ such that the particle positioned at t=0 in $a \in S(0)$ is at the time t at : $q = \widehat{f}(a,t) \in S(t)$. Of course the function $\widehat{f}$ depends itself on all the initial values, and is part of the variables to be entered in the model. This is a strong assumption indeed, as there is only one function for the whole system. The trajectory of the particle k is : $q_k(t) = \widehat{f}(a_k,t)$ with the constant a_k and its relative velocity is : $v_k = \frac{\partial \widehat{f}(a_k,t)}{\partial t}$.

Let us precise a key point : the value of the variables and their derivatives are taken in $q = \widehat{f}(a,t)$, so :

$z^i = z^i \left(\widehat{f}(a,t) \right), \partial_\alpha z^i = \partial_\alpha z^i (q) = \frac{\partial z^i}{\partial x^\alpha}(q) \neq \frac{\partial z^i}{\partial \widehat{f}} \frac{\partial \widehat{f}}{\partial x^\alpha}.$

With these assumptions the action reads :

$S_M = \sum_{k=1}^N \int_0^T L_M \left(\frac{\partial \widehat{f}(a_k,t)}{\partial t}, z^i \left(\widehat{f}(a_k,t) \right), z^i_\alpha \left(\widehat{f}(a_k,t) \right) \right) dt$

5) S(0) is a relatively compact riemanian manifold, so it is geodesically complete and there is a unique Green function $\widetilde{N} : S(0) \times S(0) \to \mathbb{R}$ such that :

$\forall u \in L^2\left(S(0)\right), u|_{\partial S(0)} = 0:$

$\int_{S(0)} \left(\Delta_a \widetilde{N}(y,a)\right) u\left(a\right) \varpi_3\left(a\right) = \int_{S(0)} \widetilde{N}(y,a) \Delta u\left(x\right) \varpi_3\left(a\right) = -u(y)$

where ϖ_3 is the induced euclidian metric on S(0) and $\Delta = div \circ \nabla$ is the Laplace-Beltrami operator on S(0), incorporating ϖ_3 (Grigor'yan [6]). $\widetilde{N}$ is smooth outside x=y and belongs to L^2(S(0)). It is symmetric : $\widetilde{N}(a,y) = \widetilde{N}(y,a)$,positive on S(0) and null on $\partial S\left(0\right)$. If S(0) is not too exotic $\widetilde{V}$ is proportional to $\int_{d(a,y)}^{\infty} \frac{sds}{V(a,s)}$ where V(a,s) is the volume of the geodesic ball centered in x. $\widetilde{N}$ is fully defined by S(0) and the induced metric on S(0). So it is a fixed function in our problem.

The quantity $\phi\left(y\right) = \int_0^T L_M \left(\frac{\partial \widehat{f}(y,t)}{\partial t}, z^i\left(\widehat{f}(y,t)\right), z^i_\alpha\left(\widehat{f}(y,t)\right)\right) dt$ is a function of $y \in S(0)$, null on $\partial S(0)$ if there is no particle on the rim at t=0, and we can reasonably assume that $\phi\left(x\right) \in L^2\left(S\left(0\right)\right)$. So we can write :

$-\phi\left(y\right) = \int_{S(0)} \Delta_x \widetilde{N}\left(y,a\right) \phi\left(a\right) \varpi_3\left(a\right)$

$-\phi\left(a_k\right) = \int_{S(0)} \Delta_x \widetilde{N}\left(a_k,a\right) \phi\left(a\right) \varpi_3\left(a\right)$

$S_M = -\sum_{k=1}^N \int_{S(0)} \left(\Delta_x \widetilde{N}\left(a_k,a\right)\right) \phi\left(a\right) \varpi_3\left(a\right) =$

$-\sum_{k=1}^N \int_{S(0)} \left(\Delta_x \widetilde{N}\left(a_k,a\right)\right) \int_0^T L_M \left(\frac{\partial \widehat{f}(a,t)}{\partial t}, z^i\left(\widehat{f}(a,t)\right), z^i_\alpha\left(\widehat{f}(a,t)\right)\right) dt \varpi_3\left(a\right)$

6) Ω has the structure of a fiber bundle with base $\mathbb{R}$ and typical fiber S(0), with trivialization : $\varphi_\Omega : S(0) \times [0,T] \to \Omega :: m = \varphi_\Omega(a,t)$

So for each $(a,t) \in S(0) \times t$ the map φ_Ω defines a point in Ω, and conversely for each $m \in \Omega$ there are unique coordinates $(a,t) \in S(0) \times t$. Indeed this is exactly how we have defined the chart. It works because the fiber bundle is trivial.

Let N be the function :

$N : \Omega \to \mathbb{R} :: N\left(\varphi_\Omega\left(a,t\right)\right)) = -\sum_k \Delta_{S(0)} \widetilde{N}\left(a_k, \varphi_\Omega\left(a,0\right)\right)) = -\sum_k \Delta_x \widetilde{N}\left(a_k,a\right)$

.

It is defined on Ω, constant for all t, smooth and null on $\partial\Omega$. For a given system N should be fully known : it is included in the initial values package.

Let be the map : $\widetilde{f} : \Omega \to \Omega :: \widetilde{f}\left(\varphi_\Omega\left(a,t\right)\right)) = \widehat{f}\left(a,t\right)$. It is defined on Ω . The particle present at t in m if any would have as coordinates : $m =$

$\varphi_\Omega(y,t)), y \in S(0)$. Its trajectory is : $q = \widehat{f}(y,t) = \widetilde{f}(\varphi_\Omega(y,t))) = \widetilde{f}(m)$. Its velocity in the chart is : $\frac{dq}{dt} = \frac{\partial}{\partial t}\widehat{f}(y,t) = \partial_0\widetilde{f}(\varphi_\Omega(y,t)))$. So we can define $V = \partial_0\widetilde{f}(m)$. We will denote : $z^0 = \widetilde{f} \Rightarrow V = \partial_0\widetilde{f} = z_0^0$ (but z^0 itself cannot be explicitly in the lagrangian) so the action reads :

$S_M =$

$\int_{S(0)} N(\varphi_\Omega(a,t))) \int_0^T L_M\left(z^i\left(\widetilde{f}(\varphi_\Omega(a,t)))\right), z_\alpha^i\left(\widetilde{f}(\varphi_\Omega(a,t)))\right)\right) dt\varpi_3(a)$

Ω has the structure of a fiber bundle with base $\mathbb{R}$ and typical fiber S(0). The volume measure on Ω can be expressed as : $\varpi_4 = \varpi_3 \otimes dt$ where dt is the Lebesgue measure on $\mathbb{R}$ (Lang [16] XV 6.4). Then:

$S_M = \int_\Omega N(m)L_M\left(z^i\left(\widetilde{f}(m)\right), z_\alpha^i\left(\widetilde{f}(m)\right)\right)\varpi_4(m)$ and the action can be written : $S_M = \int_\Omega N(m)\widetilde{f}^*L_M(z^i, z_\alpha^i)\varpi_4$

Remark : V is a vector field on Ω, which gives the velocity of the particle that would be located at the same point. Thus we have a strong analogy with a fluid mechanics model, but there are 2 differences. The V vector field is "virtual" in the meaning that I do not assume that there is a particle present : the lagrangian depends on $\widetilde{f}, N$ and this has important consequences. I do not assume any continuity of variables such as density, and any conservation law should be deduced from the model.

7) Let us now consider the creation and annihiliation of particles. The presence of a particle is felt in the action through L_M. E_M is a vector bundle and 0 is a legitimate value for ψ. After some adjustment if necessary, it is possible to guarantee that $L_M = 0$ whenever $\psi = 0$. The previous construction stands with a variable number of particles and a map $\widehat{f}$ defined over all S(0) with the convention that : $q = \widehat{f}(a,t)$ corresponds to the trajectory of a particle iff $\psi\left(\widehat{f}(a,t)\right) \neq 0$.

We have then a unique section $\psi : \Omega \to E_M$ for all of Ω and the particles. The continuity of such a section is questionable, if the particles can be created and annihiliated, or change their physical characteristics, but of course that is one of the main pending issues.

The physical vacuum means the absence of particles, that is $\psi \equiv 0$, and the action is then restricted to S_F. But the model stands as long as S(0) is defined, with any function $N(\varphi_\Omega(\xi,t))) = -\int_{S(0)} \Delta_{S(0)}\widetilde{N}(x, \varphi_\Omega(a,0))) p(a)\varpi_3(a)$ where p(a) is an arbitrary function. One gets a section ψ which is not necessarily null and defines some "fundamental state". We will come back later on this point.

4.2 The configuration bundle

1) In the following we keep the variables :
$\psi \in \Lambda_0(E_v), G \in \Lambda_1(TV; o(3,1)), \grave{A} \in \Lambda_1(TV; T_1^c U), \Theta \in \Lambda_1(TV; G_M)$,
which are sections of the respective bundles, and the map $\widetilde{f} \in C^\infty(\Omega; \Omega)$.
When it is useful we will denote : $Z_i^\Diamond = \widetilde{f}^* Z_i, L_M^\Diamond = \widetilde{f}^* L_M$.
The action is :
$S = S_M + S_F$
$S_M = \int_\Omega N L_M(V^{\Diamond\alpha}, \operatorname{Re}\psi^{\Diamond ij}, \operatorname{Im}\psi^{\Diamond ij}, \operatorname{Re}\partial_\alpha\psi^{\Diamond ij}, \operatorname{Im}\partial_\alpha\psi^{\Diamond ij}, \operatorname{Re}\grave{A}_\alpha^{\Diamond a},$
$\operatorname{Im}\grave{A}_\alpha^{\Diamond a}, \operatorname{Re}\partial_\alpha\grave{A}_\beta^{\Diamond a}, \operatorname{Im}\partial_\alpha\grave{A}_\beta^{\Diamond a}, G_\alpha^{\Diamond a}, \partial_\alpha G_\beta^{\Diamond a}, O_k'^{\Diamond\alpha}, \partial_\beta O_k'^{\Diamond\alpha})\varpi_4$
$S_F = \int_\Omega L_F\left(\operatorname{Re}\grave{A}_\alpha^a, \operatorname{Im}\grave{A}_\alpha^a, \operatorname{Re}\partial_\alpha\grave{A}_\beta^a, \operatorname{Im}\partial_\alpha\grave{A}_\beta^a, G_\alpha^a, \partial_\alpha G_\beta^a, O_k'^\alpha, \partial_\beta O_k'^\alpha\right)\varpi_4$

2) There are 16m+36 variables
$z^i = \left(V^\alpha, \operatorname{Re}\psi^{ij}, \operatorname{Im}\psi^{ij}, \operatorname{Re}\grave{A}_\alpha^a, \operatorname{Im}\grave{A}_\alpha^a, G_\alpha^a, O_k'^\alpha\right)$,
all real valued scalar functions. Let JZ be the vector bundle based over Ω modelled on the vector space spanned by (z^i) with the trivialization $Z = \varphi_{JZ}(x^\alpha, z^i)$. We will denote a vector of the tangent vector space to JZ : $v^\alpha\delta_\alpha + w^i\delta_i$.

A section of JZ is a map : $Z : \Omega \to JZ :: Z(m) = \varphi_{JZ}(x^\alpha, z^i(m))$. The first jet extension J^1Z of JZ is the set of the equivalence classes of the sections on JZ with the same first order partial derivative (Kolàr [14] IV). J^1Z is coordinated by $j^1Z = (x^\alpha, z^i, z^i_\alpha)$ and $J^1Z \to JZ$ is an affine bundle based over JZ. J^1Z is identical to the set $JZ \otimes TM^* = L(TM; JZ)$.A section of J^1Z is a map : $j^1Z(m) = \varphi_{J^1Z}(x^\alpha, z^i(m), \partial_\alpha z^i(m))$

The configuration of the system is defined by a section j^1Z of J^1Z and a map $\widetilde{f} : \Omega \to \Omega$.

With these notations the action reads :

$$\mathbf{S} = \mathbf{S}_M + \mathbf{S}_F;\ \mathbf{S}_M = \int_\Omega \mathbf{N}\widetilde{f}^*\mathbf{j}^1\mathbf{Z}^*\mathbf{L}_M\left(z^i, z^i_\alpha\right)\varpi_4;\ \mathbf{S}_F = \int_\Omega \mathbf{j}^1\mathbf{Z}^*\mathbf{L}_F\left(z^i, z^i_\alpha\right)\varpi_4 \tag{16}$$

We will denote :
$\mathcal{L}_M = N(m)\widetilde{f}^* j^1Z^*\left(L_M\left(z^i, z^i_\alpha\right)(\det O')\right)$
$\mathcal{L}_F = j^1Z^*\left(L_F\left(z^i, z^i_\alpha\right)(\det O')\right)$
So $\mathcal{L} = \mathcal{L}_M^\diamond + \mathcal{L}_F$ and $S_M = \int_\Omega \mathcal{L}_M\varpi_0; S_F = \int_\Omega \mathcal{L}_F\varpi_0$

3) The lagrangians L_M, L_F are real scalar functions on J^1Z , which together with the volume form ϖ_4 define a 4-form on Ω.

The specification of the lagrangian is a major issue in field theories. The main road to set it out is by using the constraints imposed by covariance and gauge equivariance.

The equations derived from the principle of least action lead to solutions which shall be equivariant under a gauge transformation : observers with different referentials shall be able to compare their results if they know how to pass from one referential to the other. A general theorem states that this is achieved if the lagrangian is invariant under a gauge transformation (Giachetta [5] p.70). This will give us a first batch of relations to be met by the lagrangian.

Covariance derives from the condition that the solutions, and thus the mathematical objects involved in the model, should transform as expected in a general change of chart on the manifold M : their coordinates are representative of intrinsic geometrical objects. The action is the integral of a 4-form over Ω. So the functions NL_M, L_F must be invariant under a change of chart.

We will address successively these two requirements.

We will prove that any lagrangian meeting the gauge and covariance conditions must be of the form :

$\mathbf{L} = (NL_M + L_F)\det \mathbf{O}'$

with

$\mathbf{L}_M = \mathbf{L}_M\left(V^\alpha, \operatorname{Re}\psi\Diamond^{ij}, \operatorname{Im}\psi\Diamond^{ij}, \operatorname{Re}\nabla_\alpha\psi\Diamond^{ij}, \operatorname{Im}\nabla_\alpha\psi\Diamond^{ij}, G_\alpha^{a\Diamond}, O_\alpha^{\prime\Diamond i}, \partial_\beta O_\alpha^{\prime\Diamond i}\right)$

$\mathbf{L}_F = \mathbf{L}_F\left(\operatorname{Re}\mathcal{F}^a_{A\alpha\beta}, \operatorname{Im}\mathcal{F}^a_{A\alpha\beta}, \mathcal{F}^a_{B\alpha\beta}, G^a_\alpha, O_\alpha^{\prime i}, \partial_\beta O_\alpha^{\prime i}\right)$

Furthermore G does not appear explicitly if the lagrangian does not depend on $\partial_\beta O_\alpha^{\prime i}$.

We will prove that some of the partial derivatives of the lagrangian transform as composants of tensors, thus they will be essential in definining "Noether currents".

But in the proof we will encounter many other mathematical quantities which will be useful later.

5 GAUGE EQUIVARIANCE

Any physical measurement is done by the co-ordinated network of observers. So it is not sufficient to know how each of them sets up its own apparatus,

we need to know how this set up changes as we move along the observers. A gauge transformation is thus a continuous, and we will assume a diffentiable, map : s(m)xu(m) on Q_M extended to the bundles over M by use of the gauge transformations rules. In fact the latter lead to parametrize the gauge transformations by vectors of the Lie algebra, meaning using the fiber bundle structure F_M and its 1-jet extension introduced previously. This does not concern z^0, V.

5.1 Gauge transformations on Q_M

1) From a mathematical point of view a gauge transformation is a map : $J : Q_M \to Q_M :: J(q) = q \cdot f(q)$ where $\cdot$ stands for the right action on Q_M and $f : Q_M \to (Spin(3,1) \times U)$ is such that : $f(q \cdot (s,u)) = (s,u)^{-1} \times f(q) \times (s,u)$. Applying this formula to the section $q(m) = \varphi_Q(m, 1_{Spin} \times 1_U)$ gives : $f((\varphi_Q(m,(s,u))) = (s,u)^{-1} \times f(\varphi_Q(m, 1_{Spin} \times 1_U)) \times (s,u)$

A gauge transformation can thus be equivalently defined by a map :

$j : \Omega \to (Spin(3,1) \times U) : (j_S(m), j_U(m)) = f(\varphi_Q(m, 1_{Spin} \times 1_U))$

with the action on Q_M :

$J(\varphi_Q(m,(s,u)))$

$= \varphi_Q(m,(s,u)) \cdot (s,u)^{-1}(j_S(m), j_U(m))(s,u) = \varphi_Q(m,(j_S(m)s, j_U(m)u))$

2) With pointwise product the set of gauge transformations has a group structure (the gauge group). Among all these transformations we consider those which form 1-parameter groups : the subsets of the gauge group parametrized by a real scalar τ and such that :

$J_\tau \circ J_{\tau'} = J_{\tau+\tau'} \Leftrightarrow$

$\varphi_Q(m,(j_{\tau,S}(m)s, j_{\tau,A}(m)u)(j_{\tau',S}(m)s, j_{\tau',A}(m)u))$

$= \varphi_Q(m,(j_{\tau+\tau',S}(m)s, j_{\tau+\tau',A}(m)u))$

This condition is met with $(j_{\tau,S}(m), j_{\tau,A}(m)) = \left(\exp\tau\overrightarrow{\kappa}(m), \exp\tau\overrightarrow{\theta}(m)\right)$ where $\left(\overrightarrow{\kappa}(m), \overrightarrow{\theta}(m)\right)$ is a map $\Omega \to (o(3,1), T_1^c U)$.

So we will focus on the gauge transformations such that :

$J_\tau(\varphi_Q(m,(s,u))) = \varphi_Q(m,(s,u)) \cdot \left(s^{-1}(\exp\tau\overrightarrow{\kappa}(m))s, u^{-1}\left(\exp\tau\overrightarrow{\theta}(m)\right)u\right)$

$= \varphi_Q(m,(s,u)) \cdot \left(\exp\tau Ad_{s^{-1}}\overrightarrow{\kappa}(m), \exp\tau Ad_{u^{-1}}\overrightarrow{\theta}(m)\right)$

$\frac{d}{d\tau}J_\tau(\varphi_Q(m,(s,u)))|_{\tau=0} =$

$\left([Ad_{s^{-1}}]^a_b \kappa^b(m)\widehat{\kappa}_a(q(\varphi_Q(m,(s,u))), [Ad_{h^{-1}}]^a_c \widehat{\theta}_c(m)\theta_b(q(\varphi_Q(m,(s,u))))\right)$

where $\widehat{\kappa}_a(q), \widehat{\theta}_b(q)$ are the fundamental vectors of Q_v

with $\overrightarrow{\kappa}(m) = \sum_a \kappa^a(m)\widehat{\kappa}_a(m)$, $\overrightarrow{\theta}(m) = \sum_a \theta^a(m)\widehat{\theta}_a(m)$.

Thus we have $\pi_Q(J_\tau(q)) = \pi_Q(q)$; $\frac{d}{d\tau}(J_\tau(q))|_{\tau=0} = Y(q)$ where Y is the equivariant vector field on Q_M :

$Y \in \Lambda_0(TQ_M) : Y(q) = \left(\kappa^a(q)\widehat{\kappa}_a(q), \theta^b(q)\widehat{\theta}_b(q)\right)$

with : $\left(\kappa^a(\varphi_Q(m,(s,u))), \theta^b(\varphi_Q(m,(s,u)))\right) = \left([Ad_{s^{-1}}]^a_b \kappa^b(m), [Ad_{u^{-1}}]^a_c \theta^c(m)\right)$

Y is the infinitesimal generator of J_τ : $J_\tau(q) = \exp \tau Y(q)$. Equivariant vector fields on Q_M are described in the vector bundle F_M :

$\left(\varphi_Q(m,(1,1)), \left(\widehat{\kappa}(m), \widehat{\theta}(m)\right)\right) \simeq \left(\varphi_q\left(m,(s,u)^{-1}\right), \left(Ad_s\widehat{\kappa}(m), Ad_u\widehat{\theta}(m)\right)\right)$

3) We have similar results for the gauge transformations on the fiber bundle $J^1F_M \to F_M$. A one parameter group of gauge transformations is such that : $J_\tau\left(G,\grave{A}\right) \circ J_{\tau'}\left(G,\grave{A}\right) = J_{\tau+\tau'}\left(G,\grave{A}\right)$ which is met by

$J_\tau\left(G,\grave{A}\right) = \left(Ad_{\exp\tau\overrightarrow{\kappa}}\left(G_\alpha - \tau\partial_\alpha\overrightarrow{\kappa}\right), Ad_{\exp\tau\overrightarrow{\theta}}\left(\grave{A}_\alpha - \tau\partial_\alpha\overrightarrow{\theta}\right)\right)$

$\frac{d}{d\tau}J_\tau(g(m), \grave{A}(m))|_{\tau=0}$

$= (\left[\frac{d}{d\tau}\exp\tau\overrightarrow{\kappa}, G_\alpha - \tau\partial_\alpha\overrightarrow{\kappa}\right] + \frac{d}{d\tau}\left(G_\alpha - \tau\partial_\alpha\overrightarrow{\kappa}\right),$

$\left[\frac{d}{d\tau}\exp\tau\overrightarrow{\theta}, \grave{A}_\alpha - \tau\partial_\alpha\overrightarrow{\theta}\right] + \frac{d}{d\tau}\left(\grave{A}_\alpha - \tau\partial_\alpha\overrightarrow{\theta}\right))|_{\tau=0}$

$= \left([\overrightarrow{\kappa}(m), G_\alpha] - \partial_\alpha\overrightarrow{\kappa}, \left[\overrightarrow{\theta}(m), \grave{A}_\alpha\right] - \partial_\alpha\overrightarrow{\theta}\right)$

$= \left(\left(G^a_{bc}\kappa^b(m)G^c_\alpha - \partial_\alpha\kappa^a\right)\widehat{\kappa}_a, \left(C^a_{bc}\theta^b(m)\grave{A}^c_\alpha - \partial_\alpha\theta^a\right)\widehat{\theta}_a\right)$

where the brackets are on the respective Lie algebras.

Thus this kind of gauge transformation can be parametrized in Q_M and in F_M by a section of J^1F_M :

$\zeta : \Omega \to J^1F_M :: \zeta(m) = \left(\kappa^a(m), \theta^b(m), \partial_\alpha\kappa^a(m), \partial_\alpha\theta^b(m)\right)$.

4) The gauge transformations induced by κ act on **G** which is real valued. But À is c-valued, so to be consistent one must allow θ to be c-valued, and extend T_1U to its complexified as well as the representation (W,χ) to a representation of U_c. θ and its partial derivatives are then complex valued and $\exp\tau\theta$ is well defined.

It is clear that by proceeding this way one addresses specific gauge transformations (only those that can be represented by one parameter group), and so one does not cover all the constraints on the lagrangian.

5.2 Gauge transformations in $\mathbf{J}^1 Z$

The next step is to describe how these gauge transformations act on the configuration space. At first we will describe the diffeomorphisms on $J^1 Z$, as it is a prerequisite for variational calculus.

1) A fibered isomorphism $\Phi : JZ \to JZ$ is such that there is an isomorphism $\Phi_0 : \Omega \to \Omega$ with $\pi \circ \Phi = \Phi_0 \circ \pi$ where π is the projection $JZ \to M$. Its extension $j^1\Phi : J^1 Z \to J^1 Z$ is defined by : $j^1\Phi\left(j^1_m z\right) = j^1_{\Phi_0(m)}\left(\Phi \circ z \circ \Phi_0^{-1}(m)\right)$ for any section z on JZ. A lagrangian is invariant by an automorphism $\Phi : JZ \to JZ$ if $\left(j^1\Phi\right)^* \left(\mathcal{L}\varpi_0\right) = \mathcal{L}\varpi_0$. Here ϖ_0 is the 4-form derived from a holonomic chart : $\varpi_0 = dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^0$

2) As above one focuses on one parameter groups of diffeomorphisms, with vector fields generators. A vector field on JZ is written : $Y = Y^\alpha \delta_\alpha + Y^i \delta_i$ where the basis vectors δ_α, δ_i and the components depend on the point $z \in JZ$. Y is a projectable vector field if $\pi'(z)\, Y$ is a vector field on M. Its components Y^α depend on m only. Its flow is a fibered diffeomorphism $\Phi^Y_\tau : JZ \to JZ :: \frac{d}{d\tau}\Phi^Y_\tau(z)\,|_{\tau=0} = Y(z)$ (because : $\pi\left(\Phi^Y_\tau(z)\right) = \exp\tau X\left(\pi(z)\right))$ which can be extended on $J^1 Z$ by the same procedure as above : $j^1\Phi^Y_\tau\left(j^1_m z\right) = j^1_{\Phi^X_\tau(m)}\left(\Phi^Y_\tau \circ z \circ \Phi^X_{-\tau}(m)\right)$ .The one parameter group $j^1\Phi^Y_\tau$ as for generator the vector field $j^1 Y$ on $J^1 Z$ defined by : $j^1 Y\left(j^1_m z\right) = \frac{d}{d\tau}\left(j^1\Phi^v_\tau\left(j^1_m z\right)\right)|_{\tau=0}$ for any section z on JZ. Its components are :

$j^1 Y = Y^\alpha \delta_\alpha + Y^i \delta_i + \left(\frac{\partial Y^i}{\partial \xi^\alpha} + \frac{\partial Y^i}{\partial z^j} z^j_\alpha - z^i_\beta \frac{\partial Y^\beta}{\partial \xi^\alpha}\right)\delta^\alpha_i$ (Kolàr [14] p.360).

One can write : $j^1\Phi^Y_\tau\left(j^1_m z\right) = \Phi^{j^1 Y}_\tau\left(j^1_m z\right)$

$j^1 z$ takes its value in $\mathbb{R}^{16m+32}$ and its variation is computed as :

$\Phi^{j^1 Y}_\tau\left(j^1_m z\right) - j^1_m z = \tau\left(\frac{d}{d\tau}\Phi^{j^1 Y}_\tau\left(j^1_m z\right)|_{\tau=0}\right) + \tau o(\tau) = \tau j^1 Y\left(j^1_m z\right) + \tau o(\tau)$

with $\lim_{\tau\to 0} o(\tau) = 0$

The diffeomorphism $j^1\Phi^Y_\tau$ acting on $j^1 z$ changes the value of a lagrangian $\mathcal{L}\varpi_0$: $\left(j^1 z\right)^* \mathcal{L}\varpi_0 \to \left(\Phi^{j^1 Y}_\tau\left(j^1_m z\right)\right)^* \mathcal{L}\varpi_0$

The derivative $\frac{d}{d\tau}\left(\Phi^{j^1 Y}_\tau\left(j^1_m z\right)^* \mathcal{L}\varpi_0\right)|_{\tau=0} = \left(j^1 z\right)^* \pounds_{j^1 Y}\mathcal{L}\varpi_0$ is the Lie derivative $\pounds_{j^1 Y}\mathcal{L}\varpi_0$ of $\mathcal{L}\varpi_0$ along the vector field $J^1 Y$. A lagrangian $\mathcal{L}\varpi_0$ is invariant by $j^1\Phi^Y_\tau$ iff $\pounds_{j^1 Y}\mathcal{L}\varpi_0 = 0$.

3) A one parameter group of diffeomorphism can be defined by any projectable vector field Y on JZ (this is part of the basics of variational calculus) but conversely if we are given a one parameter group of diffeomorphisms we can compute its generator. We have seen previously that a one parameter group of gauge transformations :

$q = \varphi_Q(m,(1,1)) \to \widehat{q} = \varphi_Q\left(m, \left(\exp\tau\overrightarrow{\kappa}(m), \exp\tau\overrightarrow{\theta}(m)\right)^{-1}\right)$ .on Q_M

can be parametrized by a section

$\zeta \in \Lambda_0(J^1F_M) : \zeta(m) = \left(-\kappa^a(m), -\theta^b(m), -\partial_\alpha\kappa^a(m), -\partial_\alpha\theta^b(m)\right).$

We know how such a group acts on each variable. This is a fibered map Φ^Y_τ with a vertical vector field Y (the component along δ_α is null) as generator,computed by : $Y(Z) = \frac{d}{d\tau}\Phi^Y_\tau(Z)|_{\tau=0} = \frac{d}{d\tau}\widehat{Z}(\tau)|_{\tau=0}$

We will denote :

$Y = (Y^i\delta_i)$

$= \left(Y^{\operatorname{Re}\psi^{ij}}\delta_{\operatorname{Re}\psi^{ij}}, Y^{\operatorname{Im}\psi^{ij}}\delta_{\operatorname{Im}\psi^{ij}}, Y^{G^a_\alpha}\delta_{G^a_\alpha}, Y^{\operatorname{Re}\grave{A}^a_\alpha}\delta_{\operatorname{Re}\grave{A}^a_\alpha}, Y^{\operatorname{Re}\grave{A}^a_\alpha}\delta_{\operatorname{Im}\grave{A}^a_\alpha}, Y^{O'^i_\alpha}\delta_{O'^i_\alpha}\right),$

$Y^i(Z) = \frac{d}{d\tau}\widehat{z}^i(\tau)|_{\tau=0}$

4) We have already seen the action of ζ on the potentials :

$\sum_a \grave{A}^a_\alpha\overrightarrow{\theta}_a \to \sum_a \widehat{\grave{A}}_\alpha\overrightarrow{\theta}_a = Ad_{\exp\tau\overrightarrow{\theta}}\left(\left(\sum_a \grave{A}^a_\alpha - \tau\partial_\alpha\theta^a\right)\overrightarrow{\theta}_a\right);$

$\sum_a G^a_\alpha\overrightarrow{\kappa}_a \to \sum_a \widehat{G}^a_\alpha\overrightarrow{\kappa}_a = Ad_{\mu(s)}\left(\sum_a\left(G^a_\alpha - \tau\partial_\alpha\kappa^a\right)\overrightarrow{\kappa}_a\right)$

For G :

$\frac{d}{d\tau}\left(\sum_a \widehat{G}^a_\alpha\overrightarrow{\kappa}_a\right)|_{\tau=0}$

$= \frac{d}{d\tau}\left(Ad_{\mu(s)}\left(\sum_a\left(G^a_\alpha - \tau\partial_\alpha\kappa^a\right)\overrightarrow{\kappa}_a\right)\right)|_{\tau=0} = \left(\left[\overrightarrow{\kappa}, G_\alpha\right]^a - \partial_\alpha\kappa^a\right)\overrightarrow{\kappa}_a$

$Y^{G^a_\alpha} = \frac{d}{d\tau}\left(\widehat{G}^a_\alpha\right)|_{\tau=0} = \left[\overrightarrow{\kappa}, G_\alpha\right]^a - \partial_\alpha\kappa^a = G^a_{bc}\kappa^b G^c_\alpha - \partial_\alpha\kappa^a$

For À we must compute the real and imaginary parts of the components (the structure coefficients C^b_{ac} are real) :

$\frac{d}{d\tau}\operatorname{Re}\widehat{A}^a_\alpha|_{\tau=0} = C^a_{bc}\left(\operatorname{Re}\theta^b\operatorname{Re}\acute{A}^c_\alpha - \operatorname{Im}\theta^b\operatorname{Im}\acute{A}^c_\alpha\right) - \operatorname{Re}\partial_\alpha\theta^a$

$\frac{d}{d\tau}\operatorname{Im}\widehat{A}^a_\alpha|_{\tau=0} = C^a_{bc}\left(\operatorname{Re}\theta^b\operatorname{Im}\acute{A}^c_\alpha + \operatorname{Im}\theta^b\operatorname{Re}\acute{A}^c_\alpha\right) - \operatorname{Im}\partial_\alpha\theta^a$

$Y^{\operatorname{Re}\grave{A}^a_\alpha} = \left(C^a_{bc}\left(\operatorname{Re}\theta^b\operatorname{Re}\acute{A}^c_\alpha - \operatorname{Im}\theta^b\operatorname{Im}\acute{A}^c_\alpha\right) - \operatorname{Re}\partial_\alpha\theta^a\right)$

$Y^{\operatorname{Im}\grave{A}^a_\alpha} = \left(C^a_{bc}\left(\operatorname{Re}\theta^b\operatorname{Im}\acute{A}^c_\alpha + \operatorname{Im}\theta^b\operatorname{Re}\acute{A}^c_\alpha\right) - \operatorname{Im}\partial_\alpha\theta^a\right)$

For O' :

$O'^i_\alpha \to \widehat{O'}^i_\alpha = \left[\left(\mu\left(\exp\tau\overrightarrow{\kappa}\right)\right)\right]^i_j O'^j_\alpha$

$\frac{d}{d\tau}\widehat{O'}^i_\alpha|_{\tau=0} = \left[\jmath'\circ\mu'(1)\overrightarrow{\kappa}\right]^i_j O'^j_\alpha = \left[\widetilde{\kappa}\right]^i_j O'^j_\alpha$

$Y^{O'^i_\alpha} = \left(\sum_a \kappa^a \left[\widetilde{\kappa}_a\right]^i_j O'^j_\alpha\right)$

5) The variables in L_M depend on $\widetilde{f}$. The gauge transformation acts with the values of the parameters as they are at the point $\widetilde{f}(m) : \kappa^\Diamond(m) = \kappa \circ \widetilde{f}(m), \theta^\Diamond(m) = \theta \circ \widetilde{f}(m)$ and we have for $\psi^{ij}, \grave{A}^a_\alpha, G^a_\alpha, O'^i_\alpha$:

$Z\left(\widetilde{f}(m)\right) \to \widehat{Z}\left(\widetilde{f}(m)\right) = \Phi^Y_\tau\left(Z\left(\widetilde{f}(m)\right)\right)$

$\Rightarrow Y\left(Z\left(\widetilde{f}(m)\right)\right) = \frac{d}{d\tau}\Phi^Y_\tau\left(Z\left(\widetilde{f}(m)\right)\right)|_{\tau=0}$

So the previous formulas stand if we consider the values $\widetilde{f}Z = Z^\diamond$

For ψ :

$\psi^{ij\diamond} \to \widehat{\psi^{ij\diamond}} = \sum_{kl}\left[\rho\left(\exp\tau\overrightarrow{\kappa}^\Diamond\right)\right]^i_k\left[\chi\left(\exp\tau\overrightarrow{\theta}^\Diamond\right)\right]^j_l\psi^{\Diamond kl}$

$\frac{d}{d\tau}\widehat{\psi^\diamond}|_{\tau=0} = \left(\rho'\circ\Upsilon'(1)(\overrightarrow{\kappa}^\diamond)\otimes 1 + 1\otimes\chi'(1)\left(\overrightarrow{\theta}^\diamond\right)\right)\psi^\diamond$

$\frac{d}{d\tau}\widehat{\psi^{\Diamond ij}}|_{\tau=0} = \left[\kappa^\diamond\right]^i_k\psi^{\diamond kj} + \left[\theta^\diamond\right]^j_l\psi^{\diamond il}$

$\frac{d}{d\tau}\operatorname{Re}\widehat{\psi^{\Diamond ij}}|_{\tau=0}$

$= \kappa^{\Diamond a}\operatorname{Re}\left(\left[\kappa_a\right]^i_k\psi^{\Diamond kj}\right) + \operatorname{Re}\theta^{\Diamond a}\operatorname{Re}\left(\left[\theta_a\right]^j_k\psi^{\Diamond ik}\right) - \operatorname{Im}\theta^{\Diamond a}\operatorname{Im}\left(\left[\theta_a\right]^j_k\psi^{\Diamond ik}\right)$

$\frac{d}{d\tau}\operatorname{Im}\widehat{\psi^{\Diamond ij}}|_{\tau=0}$

$= \kappa^{\Diamond a}\operatorname{Im}\left(\left[\kappa_a\right]^i_k\psi^{\Diamond kj}\right) + \operatorname{Re}\theta^{\Diamond a}\operatorname{Im}\left(\left[\theta_a\right]^j_k\psi^{\Diamond ik}\right) + \operatorname{Im}\theta^{\Diamond a}\operatorname{Re}\left(\left[\theta_a\right]^j_k\psi^{\Diamond ik}\right)$

$Y^{\operatorname{Re}\psi^{ij}}$

$= \sum_a\kappa^{\Diamond a}\operatorname{Re}\left(\left[\kappa_a\right]\left[\psi^\diamond\right]\right)^i_j + \operatorname{Re}\theta^{\Diamond a}\operatorname{Re}\left(\left[\psi^\Diamond\right]\left[\theta_a\right]^t\right)^i_j - \operatorname{Im}\theta^{\Diamond a}\operatorname{Im}\left(\left[\psi^\Diamond\right]\left[\theta_a\right]^t\right)^i_j$

$Y^{\operatorname{Im}\psi^{ij}}$

$= \sum_a\kappa^{\Diamond a}\operatorname{Im}\left(\left[\kappa_a\right]\left[\psi^\diamond\right]\right)^i_j + \operatorname{Re}\theta^{\Diamond a}\operatorname{Im}\left(\left[\psi^\Diamond\right]\left[\theta_a\right]^t\right)^i_j + \operatorname{Im}\theta^{\Diamond a}\operatorname{Re}\left(\left[\psi^\Diamond\right]\left[\theta_a\right]^t\right)^i_j$

6) The vector Y has an extention on $J^1Z : j^1Y = (Y, \partial_\beta Y)$ parametrized by $j^2\zeta = \left(\zeta, \zeta^a_\alpha, \zeta^a_{\alpha\beta}\right) \in \Lambda_0 J^2 F_M$. Its components can be computed from the general formula above, but here a direct approach is easier. Let Φ^Y_τ be the one parameter group generated by Y. It is a vertical vector, so $\Phi_0 = Id$ and its extension is for a section Z :$j^1\Phi^Y_\tau\left(j^1_m Z\right) = j^1_m\left(\Phi^Y_\tau\circ Z\right) = j^1_m\left(\widehat{Z}(\tau)\right)$. So the components $\partial_\beta z$ are computed by the partial derivatives of the components of Y :

$Y^i\left(Z(m), \zeta(m)\right) = \frac{d}{d\tau}\left(\widehat{z}^i(m,\tau)\right)|_{\tau=0}$

$Y^i_\alpha = \frac{d}{d\tau}\left(\partial_\alpha\widehat{z}^i(m,\tau)\right)|_{\tau=0} = \partial_\alpha\left(\frac{d}{d\tau}\left(\widehat{z}^i(m,\tau)\right)|_{\tau=0}\right) = \partial_\alpha Y^i\left(Z(m), \zeta(m)\right)$

For the variables depending on $\widetilde{f}$ the evaluation is still done in $\widetilde{f}(m)$ so the partial derivatives of $\widetilde{f}$ are discarded.

a) For À : $\partial_\alpha \widehat{\grave{A}}_\beta$

$= \partial_\alpha \left(Ad_u \left(\grave{A}_\beta - \tau \partial_\beta \theta \right) \right)$

$= \left(\partial_\alpha \left(Ad_u \right) \right) \left(\grave{A}_\beta - \partial_\beta \theta \right) + Ad_u \left(\partial_\alpha \grave{A}_\beta - \partial_{\alpha\beta} \theta \right)$

$= Ad_u \left[\tau \partial_\alpha \theta, \grave{A}_\beta - \tau \partial_\beta \theta \right] + Ad_u \left(\partial_\alpha \grave{A}_\beta - \tau \partial_{\alpha\beta} \theta \right)$

with the general formula : $\partial_\alpha \left(Ad_{\exp \tau \overrightarrow{\theta}} \right) = Ad_{\exp \tau \overrightarrow{\theta}} \circ ad \left(\partial_\alpha \left(\tau \theta \right) \right)$

$\partial_\alpha \widehat{\grave{A}}_\beta = Ad_{\exp \tau \overrightarrow{\theta}} \left(\left[\tau \partial_\alpha \theta, \grave{A}_\alpha - \tau \partial_\beta \theta \right] + \partial_\alpha \grave{A}_\alpha - \tau \partial_{\alpha\beta} \theta \right)$

$\frac{d}{d\tau} \partial_\alpha \widehat{A}_\beta |_{\tau=0} = \left[\overrightarrow{\theta}, \partial_\alpha \grave{A}_\beta \right] + \left[\overrightarrow{\partial_\alpha \theta}, \grave{A}_\beta \right] - \partial_{\alpha\beta} \theta$

$\frac{d}{d\tau} \partial_\alpha \widehat{A}^a_\beta |_{\tau=0} = C^a_{bc} \theta^b \partial_\alpha \grave{A}^c_\beta + C^a_{bc} \partial_\beta \theta^b \grave{A}^c_\beta - \partial_{\alpha\beta} \theta^a$

$Y_\alpha^{\operatorname{Re} \grave{A}^a_\beta} =$

$C^a_{bc} \left(\operatorname{Re} \partial_\alpha \theta^b \operatorname{Re} \grave{A}^c_\beta - \operatorname{Im} \partial_\alpha \theta^b \operatorname{Im} \grave{A}^c_\beta + \operatorname{Re} \theta^b \operatorname{Re} \partial_\alpha \grave{A}^c_\beta - \operatorname{Im} \theta^b \operatorname{Im} \partial_\alpha \grave{A}^c_\beta \right) - \operatorname{Re} \partial_{\alpha\beta} \theta^a$

$Y_\alpha^{\operatorname{Im} \grave{A}^a_\beta} =$

$C^a_{ac} \left(\operatorname{Re} \partial_\alpha \theta^b \operatorname{Im} \grave{A}^c_\beta + \operatorname{Im} \partial_\alpha \theta^c \operatorname{Re} \grave{A}^c_\beta + \operatorname{Re} \theta^b \operatorname{Im} \partial_\alpha \grave{A}^c_\beta + \operatorname{Im} \theta^b \operatorname{Re} \partial_\alpha \grave{A}^c_\beta \right) -$
$\operatorname{Im} \partial_{\alpha\beta} \theta^a$

b) For G : $\widehat{\partial_\alpha G}_\beta = Ad_{\mu(s)} \left(\left[\tau \partial_\alpha \kappa, G_\alpha - \tau \partial_\beta \kappa \right] + \partial_\alpha G_\beta - \tau \partial_{\alpha\beta} \kappa \right)$

$Y_\alpha^{G^a_\beta} = \frac{d}{d\tau} \partial_\alpha \widehat{G}^a_\beta |_{\tau=0} = \kappa^b G^a_{bc} \partial_\alpha G^c_\beta + \partial_\alpha \kappa^b G^a_{bc} G^c_\beta - \partial_{\alpha\beta} \kappa^a$

c) For O' : $\partial_\alpha \widehat{O'}^i_\beta = \left(\tau \partial_\alpha \kappa^a \left[\left(\mu \left(\exp \tau \kappa \right) \right) \right]^i_k \left[\widetilde{\kappa}_a \right]^k_j \right) O'^j_\beta + \left[\left(\mu \left(\exp \tau \kappa \right) \right) \right]^i_j \partial_\alpha O'^j_\beta$

$Y_\alpha^{O'^i_\beta} = \frac{d}{d\tau} \partial_\alpha \widehat{O'}^i_\beta |_{\tau=0} = \partial_\alpha \kappa^a \left[\widetilde{\kappa}_a \right]^i_j O'^j_\beta + \kappa^a \left[\widetilde{\kappa}_a \right]^i_j \partial_\alpha O'^j_\beta$

d) For ψ : $\partial_\alpha \widehat{\psi}^{\Diamond ij} = \sum_a (\tau \left(\partial_\alpha \kappa^a \right) \left[\rho \left(\exp \tau \kappa^\Diamond \right) \right]^i_p \left[\kappa_a \right]^p_k \left[\chi \left(\exp \tau \theta^\Diamond \right) \right]^j_l \psi^{kl}$

$+ \tau \left(\partial_\alpha \theta^a \right) \left[\rho \left(\exp \tau \kappa^\Diamond \right) \right]^i_k \left[\chi \left(\exp \tau \theta^\Diamond \right) \right]^j_p \left[\theta_a \right]^p_l \psi^{kl}$

$+ \left[\rho \left(\exp \tau \kappa^\Diamond \right) \right]^i_k \left[\chi \left(\exp \tau \theta^\Diamond \right) \right]^j_l \partial_\alpha \psi^{\Diamond kl})$

$Y_\alpha^{\operatorname{Re} \psi^{ij}} = \sum_a \kappa^{\Diamond a} \operatorname{Re} \left(\left[\kappa_a \right] \left[\partial_\alpha \psi^\Diamond \right] \right)^i_j + \partial_\alpha \kappa^{\Diamond a} \operatorname{Re} \left(\left[\kappa_a \right] \left[\partial_\alpha \psi^\Diamond \right] \right)^i_j$

$+ \operatorname{Re} \theta^{\Diamond a} \operatorname{Re} \left(\left[\partial_\alpha \psi^\Diamond \right] \left[\theta_a \right]^t \right) + \operatorname{Re} \partial_\alpha \theta^{\Diamond a} \operatorname{Re} \left(\left[\psi^\Diamond \right] \left[\theta_a \right]^t \right)^i_j$

$-\operatorname{Im}\theta^{\Diamond a}\operatorname{Im}\left(\left[\partial_\alpha\psi^{\Diamond}\right][\theta_a]^t\right)^i_j-\operatorname{Im}\partial_\alpha\theta^{\Diamond a}\operatorname{Im}\left(\left[\psi^{\Diamond}\right][\theta_a]^t\right)^i_j$
$Y_\alpha^{\operatorname{Im}\psi^{ij}}=\sum_a\kappa^{\Diamond a}\operatorname{Im}\left([\kappa_a]\left[\partial_\alpha\psi^{\Diamond}\right]\right)^i_j+\partial_\alpha\kappa^{\Diamond a}\operatorname{Im}\left([\kappa_a]\left[\partial_\alpha\psi^{\Diamond}\right]\right)^i_j$
$+\operatorname{Re}\theta^{\Diamond a}\operatorname{Im}\left(\left[\partial_\alpha\psi^{\Diamond}\right][\theta_a]^t\right)^i_j+\operatorname{Re}\partial_\alpha\theta^{\Diamond a}\operatorname{Im}\left(\left[\psi^{\Diamond}\right][\theta_a]^t\right)^i_j$
$+\operatorname{Im}\theta^{\Diamond a}\operatorname{Re}\left(\left[\partial_\alpha\psi^{\Diamond}\right][\theta_a]^t\right)^i_j+\operatorname{Im}\partial_\alpha\theta^{\Diamond a}\operatorname{Re}\left(\left[\psi^{\Diamond}\right][\theta_a]^t\right)^i_j$

5.3 Equivariance conditions

The Lie derivative of the lagrangian must be null under a gauge transformation. A direct computation gives :

$\pounds_{j^2\zeta}\mathcal{L}\varpi_0=\frac{d}{d\tau}\mathcal{L}\left(\widehat{Z}\left(j^2\zeta\right)\right)\varpi_0|_{\tau=0}$
$=\left(\sum_i\frac{\partial\mathcal{L}_M}{\partial z^i}\frac{d\widehat{z}^{\Diamond i}}{d\tau}|_{\tau=0}+\frac{\partial\mathcal{L}_M}{\partial\partial_\alpha z^i}\frac{d\partial_\alpha\widehat{z}^{\Diamond i}}{d\tau}|_{\tau=0}+\frac{\partial\mathcal{L}_F}{\partial Z^i}\frac{d\widehat{Z}^i}{d\tau}|_{\tau=0}+\frac{\partial\mathcal{L}_F}{\partial\partial_\alpha Z^i}\frac{d\partial_\alpha\widehat{Z}^i}{d\tau}|_{\tau=0}\right)\varpi_0$

By definition :

$\frac{d\widehat{z}^{\Diamond i}}{d\tau}|_{\tau=0}=\frac{d\widehat{z}^i}{d\tau}|_{\tau=0}=Y^i(Z,\zeta),\frac{d\partial_\alpha\widehat{z}^{\Diamond i}}{d\tau}|_{\tau=0}=\frac{d\partial_\alpha\widehat{z}^i}{d\tau}|_{\tau=0}=Y^i_\alpha(Z,\zeta)$

$\mathcal{L}_M$ depends on f and not $\mathcal{L}_F$, so for all $j^2\zeta$ we must have the identities :

$\sum_i\frac{\partial\mathcal{L}_M}{\partial z^i}Y^i(Z,\zeta)+\sum_\alpha\frac{\partial\mathcal{L}_M}{\partial\partial_\alpha z^i}Y^i_\alpha(Z,\zeta)=0$
$\sum_i\frac{\partial\mathcal{L}_F}{\partial z^i}Y^i(Z,\zeta)+\sum_\alpha\frac{\partial\mathcal{L}_F}{\partial\partial_\alpha z^i}Y^i_\alpha(Z,\zeta)=0$

Some of the variables can appear explicitly or through an other one. In order to avoid confusion we will use the following conventions :

$\frac{d}{dz^i},\frac{d}{d\partial_\alpha z^i}$ such as $\frac{d}{d\operatorname{Re}\psi^{ij}},\frac{d}{d\partial_\alpha\operatorname{Re}\psi^{ij}},...$ denotes the full partial derivatives with respect to the variables z^i,z^i_α

$\frac{\partial}{\partial z^i},\frac{\partial}{\partial\partial_\alpha z^i}$ such as $\frac{\partial}{\partial\operatorname{Re}\psi^{ij}},\frac{\partial}{\partial\partial_\alpha\operatorname{Re}\psi^{ij}},...$ denotes the partial derivatives with respect to the variables z^i,z^i_α only when they appear explicitly

So the previous identities read :

$\sum_i\frac{d\mathcal{L}_M}{dz^i}Y^i(Z,\zeta)+\sum_\alpha\frac{d\mathcal{L}_M}{d\partial_\alpha z^i}Y^i_\alpha(Z,\zeta)=0$
$\sum_i\frac{d\mathcal{L}_F}{dz^i}Y^i(Z,\zeta)+\sum_\alpha\frac{d\mathcal{L}_F}{d\partial_\alpha z^i}Y^i_\alpha(Z,\zeta)=0$

5.3.1 Lagrangien $\mathcal{L}_F$

We address first the lagrangian $\mathcal{L}_F$.

1)

The terms in the second order in $j^2\zeta$ give :

$$\forall\alpha,\beta,a:0=\frac{d\mathcal{L}_F}{d\partial_\alpha G^a_\beta}+\frac{d\mathcal{L}_F}{d\partial_\beta G^a_\alpha};0=\frac{d\mathcal{L}_F}{d\partial_\beta \operatorname{Re}\grave{A}^a_\alpha}+\frac{d\mathcal{L}_F}{d\partial_\alpha \operatorname{Re}\grave{A}^a_\beta}; \tag{17}$$

$$0=\frac{d\mathcal{L}_F}{d\partial_\beta \operatorname{Im}\grave{A}^a_\alpha}+\frac{d\mathcal{L}_F}{d\partial_\alpha \operatorname{Im}\grave{A}^a_\beta} \tag{18}$$

where we use : $\partial_{\beta\alpha}\kappa^b=\partial_{\alpha\beta}\kappa^b,\partial_{\beta\alpha}\theta^b=\partial_{\alpha\beta}\theta^b$

So the partial derivatives are antisymmetric in $\partial_\alpha G^a_\beta,\partial_\alpha\grave{A}^a_\beta$ as could be expected.

The terms in the first order in $\partial_\alpha\kappa^a$ give :

$$\forall a,\alpha:0=-\frac{d\mathcal{L}_F}{dG^a_\alpha}+\sum_{\beta b}\frac{d\mathcal{L}_F}{d\partial_\alpha G^b_\beta}\left[\overrightarrow{\kappa}_a,G_\beta\right]^b+\sum_{\beta i}\frac{d\mathcal{L}_F}{d\partial_\alpha O'^i_\beta}\left(\left[\widetilde{\kappa}_a\right]\left[O'\right]\right)^i_\beta \tag{19}$$

The terms in the first order in $\partial_\alpha\theta^a$ give :

$$\forall\alpha,a:0=-\frac{d\mathcal{L}_F}{d\operatorname{Re}\grave{A}^a_\alpha}+\sum_\beta\frac{d\mathcal{L}_F}{d\operatorname{Re}\partial_\alpha\grave{A}^b_\beta}\operatorname{Re}\left[\overrightarrow{\theta}_a,\grave{A}_\beta\right]^b+\frac{d\mathcal{L}_F}{d\operatorname{Im}\partial_\alpha\grave{A}^b_\beta}\operatorname{Im}\left[\overrightarrow{\theta}_a,\grave{A}_\beta\right]^b \tag{20}$$

$$\forall\alpha,a:0=-\frac{d\mathcal{L}_F}{d\operatorname{Im}\grave{A}^a_\alpha}+\sum_\beta-\frac{d\mathcal{L}_F}{d\operatorname{Re}\partial_\alpha\grave{A}^b_\beta}\operatorname{Im}\left[\overrightarrow{\theta}_a,\grave{A}_\beta\right]^b+\frac{d\mathcal{L}_F}{d\operatorname{Im}\partial_\alpha\grave{A}^b_\beta}\operatorname{Re}\left[\overrightarrow{\theta}_a,\grave{A}_\beta\right]^b \tag{21}$$

where we use the fact that $\left(\overrightarrow{\theta}_a\right)$ is a real basis and the structure coefficients C^a_{bc} are real :

$\operatorname{Re}\left[\overrightarrow{\theta}_a,\grave{A}_\beta\right]^b=\operatorname{Re}\sum_c\left(C^b_{ac}\grave{A}^c_\beta\right)=\sum_c C^b_{ac}\operatorname{Re}\grave{A}^c_\beta$

The terms in κ^a give :

$$\begin{aligned}\forall a:0=&\textstyle\sum_{\alpha\beta b}\frac{d\mathcal{L}_F}{dG^b_\alpha}\left[\overrightarrow{\kappa}_a,G_\alpha\right]^b+\frac{d\mathcal{L}_F}{d\partial_\beta G^b_\alpha}\left[\overrightarrow{\kappa}_a,\partial_\beta G_\alpha\right]^b\\&\textstyle+\sum_{\alpha\beta i}\frac{d\mathcal{L}_F}{dO'^i_\alpha}\left(\left[\widetilde{\kappa}_a\right]\left[O'\right]\right)^i_\alpha+\frac{d\mathcal{L}_F}{d\partial_\beta O'^i_\alpha}\left(\left[\widetilde{\kappa}_a\right]\left[\partial_\beta O'\right]\right)^i_\alpha\end{aligned} \tag{22}$$

The terms in θ^a give :

$$\forall a : 0 = \sum_{\alpha b} \frac{d\mathcal{L}_F}{d\,\mathrm{Re}\,\grave{A}^b_\alpha} \mathrm{Re}\left[\overrightarrow{\theta}_a, \grave{A}_\alpha\right]^b + \frac{d\mathcal{L}_F}{d\,\mathrm{Im}\,\grave{A}^b_\alpha} \mathrm{Im}\left[\overrightarrow{\theta}_a, \grave{A}_\alpha\right]^b + \sum_\beta \frac{\partial\mathcal{L}_F}{\partial\,\mathrm{Re}\,\partial_\beta\grave{A}^b_\alpha} \mathrm{Re}\left[\overrightarrow{\theta}_a, \partial_\beta\grave{A}_\alpha\right]^b + \frac{\partial\mathcal{L}_F}{\partial\,\mathrm{Im}\,\partial_\beta\grave{A}^b_\alpha} \mathrm{Im}\left[\overrightarrow{\theta}_a, \partial_\beta\grave{A}_\alpha\right]^b \tag{23}$$

$$\forall a : 0 = \sum_{\alpha b} -\frac{d\mathcal{L}_F}{d\,\mathrm{Re}\,\grave{A}^b_\alpha} \mathrm{Im}\left[\overrightarrow{\theta}_a, \grave{A}_\alpha\right]^b + \frac{d\mathcal{L}_F}{d\,\mathrm{Im}\,\grave{A}^b_\alpha} \mathrm{Re}\left[\overrightarrow{\theta}_a, \grave{A}_\alpha\right]^b + \sum_\beta -\frac{\partial\mathcal{L}_F}{\partial\,\mathrm{Re}\,\partial_\beta\grave{A}^b_\alpha} \mathrm{Im}\left[\overrightarrow{\theta}_a, \partial_\beta\grave{A}_\alpha\right]^b + \frac{\partial\mathcal{L}_F}{\partial\,\mathrm{Im}\,\partial_\beta\grave{A}^b_\alpha} \mathrm{Re}\left[\overrightarrow{\theta}_a, \partial_\beta\grave{A}_\alpha\right]^b \tag{24}$$

2) Let be $F^a_{G\alpha\beta} = \partial_\alpha G_\beta + \partial_\beta G_\alpha$

By changing the variables :

$\partial_\alpha G_\beta \to \frac{1}{2}\left(\mathcal{F}^a_{G\alpha\beta} - G^a_{bc}G^b_\alpha G^c_\beta + F^a_{G\alpha\beta}\right)$

$\partial_\beta G_\alpha \to \frac{1}{2}\left(-\mathcal{F}^a_{G\alpha\beta} + G^a_{bc}G^b_\alpha G^c_\beta + F^a_{G\alpha\beta}\right)$

the equation 17 gives, when renaming $\mathcal{L}'_F$ the new lagrangian :

$\forall \alpha, \beta, b : \frac{d\mathcal{L}'_F}{dF^b_{G\alpha\beta}} + \frac{d\mathcal{L}'_F}{dF^b_{G\beta\alpha}} + \frac{d\mathcal{L}'_F}{d\mathcal{F}^b_{G\alpha\beta}} - \frac{d\mathcal{L}'_F}{d\mathcal{F}^b_{G\beta\alpha}} = 0$

with the reversion of α, β :

$\frac{d\mathcal{L}'_F}{dF^b_{G\beta\alpha}} + \frac{d\mathcal{L}'_F}{dF^b_{G\alpha\beta}} + \frac{d\mathcal{L}'_F}{d\mathcal{F}^b_{G\beta\alpha}} - \frac{d\mathcal{L}'_F}{d\mathcal{F}^b_{G\alpha\beta}} = 0$

and adding the two : $2\left(\frac{d\mathcal{L}'_F}{dF^b_{G\beta\alpha}} + \frac{d\mathcal{L}'_F}{dF^b_{G\alpha\beta}}\right) = 0$

$F^a_{G\alpha\beta} = F^a_{G\beta\alpha} \Rightarrow \frac{d\mathcal{L}'_F}{dF^b_{G\alpha\beta}} = \frac{d\mathcal{L}'_F}{dF^b_{G\beta\alpha}} = 0$

We have a similar calculation for À. The first result is that the partial derivatives of the potential G and À factorize through the curvature forms $\mathcal{F}_A, \mathcal{F}_G$.

$$\frac{d\mathcal{L}_F}{d\partial_\beta G^a_\alpha} = -\frac{d\mathcal{L}_F}{d\partial_\alpha G^a_\beta} = 2\frac{d\mathcal{L}_F}{d\mathcal{F}^a_{G\beta\alpha}} \tag{25}$$

$$\frac{d\mathcal{L}_F}{d\,\mathrm{Re}\,\partial_\alpha\grave{A}^a_\beta} = -\frac{d\mathcal{L}_F}{d\,\mathrm{Re}\,\partial_\beta\grave{A}^a_\alpha} = 2\frac{d\mathcal{L}_F}{d\,\mathrm{Re}\,\mathcal{F}^a_{\alpha\beta}}; \frac{d\mathcal{L}_F}{d\,\mathrm{Im}\,\partial_\alpha\grave{A}^a_\beta} = -\frac{d\mathcal{L}_F}{d\,\mathrm{Im}\,\partial_\beta\grave{A}^a_\alpha} = 2\frac{d\mathcal{L}_F}{d\,\mathrm{Im}\,\mathcal{F}^a_{\alpha\beta}} \tag{26}$$

3) Equation 19 becomes :

$\forall a, \alpha : -\frac{d\mathcal{L}_F}{dG^a_\alpha} + 2\sum_{\beta b} \frac{d\mathcal{L}_F}{d\mathcal{F}^a_{G\alpha\beta}} \left[\overrightarrow{\kappa}_a, G_\beta\right]^b + \sum_{\beta i} \frac{d\mathcal{L}_F}{d\partial_\alpha O'^i_\beta} \left(\left[\widetilde{\kappa}_a\right][O']\right)^i_\beta = 0$

But $\mathcal{F}^b_{G\lambda\mu} = \partial_\lambda G^b_\mu - \partial_\mu G^b_\lambda + G^b_{cd} G^c_\lambda G^d_\mu$ so

$\frac{d\mathcal{L}_F}{dG^a_\alpha} = \frac{\partial\mathcal{L}_F}{\partial G^a_\alpha} + \sum_{b\lambda\mu} \frac{d\mathcal{L}_F}{d\mathcal{F}^b_{G\lambda\mu}} \frac{\partial \mathcal{F}^b_{G\lambda\mu}}{\partial G^a_\alpha} = \frac{\partial\mathcal{L}_F}{\partial G^a_\alpha} + \sum_{bcd\lambda\mu} \frac{d\mathcal{L}_F}{d\mathcal{F}^b_{G\lambda\mu}} \left(G^b_{cd}\delta^a_c\delta^\alpha_\lambda G^d_\mu + G^b_{cd}G^c_\lambda\delta^a_d\delta^\alpha_\mu\right)$

$= \frac{\partial\mathcal{L}_F}{\partial G^a_\alpha} + \sum_{bcd\lambda\mu} \frac{d\mathcal{L}_F}{d\mathcal{F}^b_{G\alpha\mu}} \left(G^b_{ad}G^d_\mu\right) + \frac{d\mathcal{L}_F}{d\mathcal{F}^b_{G\lambda\alpha}} \left(G^b_{cd}G^c_\lambda\right) = \frac{\partial\mathcal{L}_F}{\partial G^a_\alpha} + 2\sum_{bc\beta} \frac{d\mathcal{L}_F}{d\mathcal{F}^b_{G\alpha\beta}} \left(G^b_{ac}G^c_\beta\right)$

$\frac{d\mathcal{L}_F}{dG^a_\alpha} = \frac{\partial\mathcal{L}_F}{\partial G^a_\alpha} + 2\sum_{b\beta} \frac{d\mathcal{L}_F}{d\mathcal{F}^b_{G\alpha\beta}} \left[\overrightarrow{\kappa}_a, G_\beta\right]^b$

Thus :

$$\forall a, \alpha : 0 = -\frac{\partial\mathcal{L}_F}{\partial G^a_\alpha} + \sum_{\beta i} \frac{d\mathcal{L}_F}{d\partial_\alpha O'^i_\beta} \left(\left[\widetilde{\kappa}_a\right][O']\right)^i_\beta \tag{27}$$

So the second result is that $\mathcal{L}_F$ does not depend explicitly on G if it does not depend on the partial derivatives $\partial_\alpha O'^i_\beta$.

Equation 20 becomes :

$\forall \alpha, a : -\frac{d\mathcal{L}_F}{d\operatorname{Re}\grave{A}^a_\alpha} + 2\sum_\beta \frac{d\mathcal{L}_F}{d\operatorname{Re}\mathcal{F}^b_{\alpha\beta}} \operatorname{Re}\left[\overrightarrow{\theta}_a, \grave{A}_\beta\right]^b + \frac{d\mathcal{L}_F}{d\operatorname{Im}\mathcal{F}^b_{\alpha\beta}} \operatorname{Im}\left[\overrightarrow{\theta}_a, \grave{A}_\beta\right]^b = 0$

But

$\operatorname{Re}\mathcal{F}^b_{A\lambda\mu} = \operatorname{Re}\partial_\lambda \grave{A}^b_\mu - \operatorname{Re}\partial_\mu \grave{A}^b_\lambda + C^b_{cd}\left(\operatorname{Re}\grave{A}^c_\lambda \operatorname{Re}\grave{A}^d_\mu - \operatorname{Im}\grave{A}^c_\lambda \operatorname{Im}\grave{A}^d_\mu\right)$

$\operatorname{Im}\mathcal{F}^b_{A\lambda\mu} = \operatorname{Im}\partial_\lambda \grave{A}^b_\mu - \operatorname{Im}\partial_\mu \grave{A}^b_\lambda + C^b_{cd}\left(\operatorname{Re}\grave{A}^c_\lambda \operatorname{Im}\grave{A}^d_\mu + \operatorname{Im}\grave{A}^c_\lambda \operatorname{Re}\grave{A}^d_\mu\right)$

with $\frac{d\mathcal{L}_F}{d\operatorname{Re}\grave{A}^a_\alpha} = \frac{\partial\mathcal{L}_F}{\partial\operatorname{Re}\grave{A}^a_\alpha} + \sum_{b\lambda\mu} \frac{d\mathcal{L}_F}{d\operatorname{Re}\mathcal{F}^b_{G\lambda\mu}} \frac{\partial\operatorname{Re}\mathcal{F}^b_{G\lambda\mu}}{\partial G^a_\alpha} + \frac{d\mathcal{L}_F}{d\operatorname{Im}\mathcal{F}^b_{G\lambda\mu}} \frac{\partial\operatorname{Im}\mathcal{F}^b_{G\lambda\mu}}{\partial G^a_\alpha}$

$\frac{d\mathcal{L}_F}{d\operatorname{Re}\grave{A}^a_\alpha} = \frac{\partial\mathcal{L}_F}{\partial\operatorname{Re}\grave{A}^a_\alpha} + 2\sum_{b\beta} \frac{d\mathcal{L}_F}{d\operatorname{Re}\mathcal{F}^b_{G\alpha\beta}} C^b_{ac}\operatorname{Re}\grave{A}^c_\beta + \frac{d\mathcal{L}_F}{d\operatorname{Im}\mathcal{F}^b_{G\alpha\beta}} C^b_{ac}\operatorname{Im}\grave{A}^c_\beta$

$\frac{d\mathcal{L}_F}{d\operatorname{Re}\grave{A}^a_\alpha} = \frac{\partial\mathcal{L}_F}{\partial\operatorname{Re}\grave{A}^a_\alpha} + 2\sum_{b\beta} \frac{d\mathcal{L}_F}{d\operatorname{Re}\mathcal{F}^b_{G\alpha\beta}} \operatorname{Re}\left[\overrightarrow{\theta}_a, \grave{A}_\beta\right]^b + \frac{d\mathcal{L}_F}{d\operatorname{Im}\mathcal{F}^b_{G\alpha\beta}} \operatorname{Im}\left[\overrightarrow{\theta}_a, \grave{A}_\beta\right]^b$

That is : $\forall \alpha, a : \frac{\partial\mathcal{L}_F}{\partial\operatorname{Re}\grave{A}^a_\alpha} = 0$

Similarly equation 21 gives :

$\forall \alpha, a : -\frac{d\mathcal{L}_F}{d\operatorname{Im}\grave{A}^a_\alpha} + 2\sum_\beta -\frac{d\mathcal{L}_F}{d\operatorname{Re}\mathcal{F}^b_{\alpha\beta}} \operatorname{Im}\left[\overrightarrow{\theta}_a, \grave{A}_\beta\right]^b + \frac{d\mathcal{L}_F}{d\operatorname{Im}\mathcal{F}^b_{\alpha\beta}} \operatorname{Re}\left[\overrightarrow{\theta}_a, \grave{A}_\beta\right]^b = 0$

$\frac{d\mathcal{L}_F}{d\operatorname{Im}\grave{A}^a_\alpha} = \frac{\partial\mathcal{L}_F}{\partial\operatorname{Im}\grave{A}^a_\alpha} + 2\sum_{b\beta} -\frac{d\mathcal{L}_F}{d\operatorname{Re}\mathcal{F}^b_{G\alpha\beta}} \operatorname{Im}\left[\overrightarrow{\theta}_a, \grave{A}_\beta\right]^b + \frac{d\mathcal{L}_F}{d\operatorname{Im}\mathcal{F}^b_{G\alpha\beta}} \operatorname{Re}\left[\overrightarrow{\theta}_a, \grave{A}_\beta\right]^b$

$\Rightarrow \frac{\partial\mathcal{L}_F}{\partial\operatorname{Im}\grave{A}^a_\alpha} = 0$

The third result is that À factorizes through the curvature form $\mathcal{F}_A$.

$$\forall \alpha, a : \frac{\partial\mathcal{L}_F}{\partial\operatorname{Re}\grave{A}^a_\alpha} = 0; \frac{\partial\mathcal{L}_F}{\partial\operatorname{Im}\grave{A}^a_\alpha} = 0 \tag{28}$$

4) Equation 22 can be written :

$\forall a: 0=\sum_{\alpha b}\left(\frac{\partial \mathcal{L}_F}{\partial G^b_\alpha}+2\sum_{\beta c}\frac{d\mathcal{L}_F}{d\mathcal{F}^c_{G\alpha\beta}}\left[\overrightarrow{\kappa}_b,G_\beta\right]^c\right)\left[\overrightarrow{\kappa}_a,G_\alpha\right]^b+2\sum_{\alpha\beta b}\frac{d\mathcal{L}_F}{d\mathcal{F}^b_{G\alpha\beta}}\left[\overrightarrow{\kappa}_a,\partial_\alpha G_\beta\right]^b$
$+\sum_{\alpha\beta i}\frac{d\mathcal{L}_F}{dO'^i_\alpha}\left([\widetilde{\kappa}_a]\,[O']\right)^i_\alpha+\frac{d\mathcal{L}_F}{d\partial_\beta O'^i_\alpha}\left([\widetilde{\kappa}_a]\,[\partial_\beta O']\right)^i_\alpha$

$\forall a: 0=2\sum_{\alpha\beta b}\frac{d\mathcal{L}_F}{d\mathcal{F}^b_{G\alpha\beta}}\left(\left[\left[\overrightarrow{\kappa}_a,G_\alpha\right],G_\beta\right]+\left[\overrightarrow{\kappa}_a,\partial_\alpha G_\beta\right]\right)^b+\sum_{\alpha b}\frac{\partial \mathcal{L}_F}{\partial G^b_\alpha}\left[\overrightarrow{\kappa}_a,G_\alpha\right]^b$
$+\sum_{ij}[\widetilde{\kappa}_a]^i_j\sum_\alpha\left(\frac{d\mathcal{L}_F}{dO'^i_\alpha}O'^j_\alpha+\sum_\beta\frac{d\mathcal{L}_F}{d\partial_\beta O'^i_\alpha}\partial_\beta O'^j_\alpha\right)$

$\sum_{\alpha\beta b}\frac{d\mathcal{L}_F}{d\mathcal{F}^b_{G\alpha\beta}}\left(\left[\left[\overrightarrow{\kappa}_a,G_\alpha\right],G_\beta\right]+\left[\overrightarrow{\kappa}_a,\partial_\alpha G_\beta\right]\right)^b$
$=\sum_{b,\alpha<\beta}\frac{d\mathcal{L}_F}{d\mathcal{F}^b_{G\alpha\beta}}\left(\left[\left[\overrightarrow{\kappa}_a,G_\alpha\right],G_\beta\right]+\left[\overrightarrow{\kappa}_a,\partial_\alpha G_\beta\right]-\left[\left[\overrightarrow{\kappa}_a,G_\beta\right],G_\alpha\right]-\left[\overrightarrow{\kappa}_a,\partial_\beta G_\alpha\right]\right)^b$

The brackets are computed in the Lie algebra. The Jacobi identities give :

$\left[\left[\overrightarrow{\kappa}_a,G_\alpha\right],G_\beta\right]+\left[\left[G_\alpha,G_\beta\right],\overrightarrow{\kappa}_a\right]+\left[\left[G_\beta,\overrightarrow{\kappa}_a\right],G_\alpha\right]=0$
$\left[\left[\overrightarrow{\kappa}_a,G_\alpha\right],G_\beta\right]-\left[\left[\overrightarrow{\kappa}_a,G_\beta\right],G_\alpha\right]=\left[\left[\overrightarrow{\kappa}_a,G_\alpha\right],G_\beta\right]+\left[\left[G_\beta,\overrightarrow{\kappa}_a\right],G_\alpha\right]=-\left[\left[G_\alpha,G_\beta\right],\overrightarrow{\kappa}_a\right]$
$\left(\left[\left[\overrightarrow{\kappa}_a,G_\alpha\right],G_\beta\right]+\left[\overrightarrow{\kappa}_a,\partial_\alpha G_\beta\right]-\left[\left[\overrightarrow{\kappa}_a,G_\beta\right],G_\alpha\right]-\left[\overrightarrow{\kappa}_a,\partial_\beta G_\alpha\right]\right)$
$=\left[\overrightarrow{\kappa}_a,\left[G_\alpha,G_\beta\right]\right]+\left[\overrightarrow{\kappa}_a,\partial_\alpha G_\beta\right]-\left[\overrightarrow{\kappa}_a,\partial_\beta G_\alpha\right]=\left[\overrightarrow{\kappa}_a,\mathcal{F}_{G\alpha\beta}\right]$
$\sum_{\alpha\beta b}\frac{d\mathcal{L}_F}{d\mathcal{F}^b_{G\alpha\beta}}\left(\left[\left[\overrightarrow{\kappa}_a,G_\alpha\right],G_\beta\right]+\left[\overrightarrow{\kappa}_a,\partial_\alpha G_\beta\right]\right)^b=\sum_{b,\alpha,\beta}\frac{d\mathcal{L}_F}{d\mathcal{F}^b_{G\alpha\beta}}\left[\overrightarrow{\kappa}_a,\mathcal{F}_{G\alpha\beta}\right]^b$

So equation 22 reads :

$$\forall a: 0=\sum_{\alpha b}\left(\frac{\partial \mathcal{L}_F}{\partial G^b_\alpha}\left[\overrightarrow{\kappa}_a,G_\alpha\right]^b+2\sum_\beta\frac{d\mathcal{L}_F}{d\mathcal{F}^b_{G\alpha\beta}}\left[\overrightarrow{\kappa}_a,\mathcal{F}_{G\alpha\beta}\right]^b\right)$$
$$+\sum_{ij}[\widetilde{\kappa}_a]^i_j\sum_\alpha\left(\frac{d\mathcal{L}_F}{dO'^i_\alpha}O'^j_\alpha+\sum_\beta\frac{d\mathcal{L}_F}{d\partial_\beta O'^i_\alpha}\partial_\beta O'^j_\alpha\right) \tag{29}$$

Remark : with $\frac{d\det O'}{dO'^i_\alpha}=O^\alpha_i\det O'$

$\sum_{\alpha\beta i}\frac{d\mathcal{L}_F}{dO'^i_\alpha}\left([\widetilde{\kappa}_a]\,[O']\right)^i_\alpha=\sum_{\alpha\beta i}(\det O')\frac{dL_F}{dO'^i_\alpha}\left([\widetilde{\kappa}_a]\,[O']\right)^i_\alpha+L_F\left([\widetilde{\kappa}_a]\,[O']\right)^i_\alpha O^\alpha_i\det O'$
$=\det O'\sum_{\alpha\beta i}\frac{dL_F}{dO'^i_\alpha}\left([\widetilde{\kappa}_a]\,[O']\right)^i_\alpha+L_F Tr\,[\widetilde{\kappa}_a]=\det O'\sum_{\alpha\beta i}\frac{dL_F}{dO'^i_\alpha}\left([\widetilde{\kappa}_a]\,[O']\right)^i_\alpha$

(the matrices $[\widetilde{\kappa}_a]$ are traceless) .So the equation 29 stands also with substituting L_F to $\mathcal{L}_F$.

5) Equation 23 becomes :

$\forall a: 0=2\sum_\beta\left(\frac{\partial L_F}{\partial\operatorname{Re}\mathcal{F}^c_{A\alpha\beta}}\operatorname{Re}\left[\overrightarrow{\theta}_b,\grave{A}_\beta\right]^c+\frac{\partial L_F}{\partial\operatorname{Im}\mathcal{F}^c_{A\alpha\beta}}\operatorname{Im}\left[\overrightarrow{\theta}_b,\grave{A}_\beta\right]^c\right)\operatorname{Re}\left[\widehat{\theta}_a,\grave{A}_\alpha\right]^b$
$+2\sum_\beta\left(-\frac{\partial L_F}{\partial\operatorname{Re}\mathcal{F}^c_{A\alpha\beta}}\operatorname{Im}\left[\overrightarrow{\theta}_b,\grave{A}_\beta\right]^c+\frac{\partial L_F}{\partial\operatorname{Im}\mathcal{F}^c_{A\alpha\beta}}\operatorname{Re}\left[\overrightarrow{\theta}_b,\grave{A}_\beta\right]^c\right)\operatorname{Im}\left[\overrightarrow{\theta}_a,\grave{A}_\alpha\right]^b$

$-2\frac{\partial L_F}{\partial \operatorname{Re}\mathcal{F}^b_{A\alpha\beta}} \operatorname{Re}\left[\overrightarrow{\theta}_a, \partial_\beta \grave{A}_\alpha\right]^b - 2\frac{\partial L_F}{\partial \operatorname{Im}\mathcal{F}^b_{A\alpha\beta}} \operatorname{Im}\left[\overrightarrow{\theta}_a, \partial_\beta \grave{A}_\alpha\right]^b$
that is :
$0 = \sum_{\alpha\beta b} \frac{\partial L_F}{\partial \operatorname{Re}\mathcal{F}^b_{G\alpha\beta}} \operatorname{Re}\left(\left[\left[\overrightarrow{\theta}_a, \grave{A}_\alpha\right], \grave{A}_\beta\right] - \left[\overrightarrow{\theta}_a, \partial_\beta \grave{A}_\alpha\right]\right)^b$
$+\frac{\partial L_F}{\partial \operatorname{Im}\mathcal{F}^b_{B\alpha\beta}} \operatorname{Im}\left(\left[\left[\overrightarrow{\theta}_a, \grave{A}_\alpha\right], \grave{A}_\beta\right] - \left[\overrightarrow{\theta}_a, \partial_\beta \grave{A}_\alpha\right]\right)^b$
and the same calculation as previously gives :

$$\forall a : 0 = \sum_{\alpha\beta b} \frac{\partial L_F}{\partial \operatorname{Re}\mathcal{F}^b_{A\alpha\beta}} \operatorname{Re}\left(\left[\overrightarrow{\theta}_a, \mathcal{F}_{A\alpha\beta}\right]\right)^b + \frac{\partial L_F}{\partial \operatorname{Im}\mathcal{F}^b_{A\alpha\beta}} \operatorname{Im}\left(\left[\overrightarrow{\theta}_a, \mathcal{F}_{A\alpha\beta}\right]\right)^b \tag{30}$$

Similar calculation with equation 24 gives :

$$\forall a : 0 = \sum_{\alpha\beta b} -\frac{\partial L_F}{\partial \operatorname{Re}\mathcal{F}^b_{A\alpha\beta}} \operatorname{Im}\left(\left[\overrightarrow{\theta}_a, \mathcal{F}_{A\alpha\beta}\right]\right)^b + \frac{\partial L_F}{\partial \operatorname{Im}\mathcal{F}^b_{A\alpha\beta}} \operatorname{Re}\left(\left[\overrightarrow{\theta}_a, \mathcal{F}_{A\alpha\beta}\right]\right)^b \tag{31}$$

5.3.2 Lagrangian $\mathcal{L}_M$

The partial derivatives of G and À appear only in the curvature forms, and these only in the interactions fields/fields, so it is legitimate to assume that L_M does not depend on these variables :
$\frac{d\mathcal{L}_M}{d\partial_\beta G^a_\alpha} = 0; \frac{d\mathcal{L}_M}{d \operatorname{Re}\partial_\alpha \grave{A}^a_\beta} = 0; \frac{d\mathcal{L}_M}{d \operatorname{Im}\partial_\alpha \grave{A}^a_\beta} = 0$
We proceed as above.

1) Terms in first order in $\partial_\alpha \kappa^a$ give :

(32)

$$\forall a, \alpha : 0 = \sum_{ij} \left(\frac{d\mathcal{L}_M}{d \operatorname{Re}\partial_\alpha \psi^{ij}} \operatorname{Re}\left([\kappa_a]\left[\psi^\diamondsuit\right]\right)^i_j + \frac{d\mathcal{L}_M}{d \operatorname{Im}\partial_\alpha \psi^{ij}} \operatorname{Im}\left([\kappa_a]\left[\psi^\diamondsuit\right]\right)^i_j - \frac{d\mathcal{L}_M}{dG^a_\alpha}\right)$$
$$+\sum_{\beta i} \frac{d\mathcal{L}_M}{d\partial_\alpha O'^i_\beta}\left([\widetilde{\kappa}_a][O']\right)^i_\beta$$

Terms in first order in $\partial_\alpha \theta^a$ give :

$$\forall a, \alpha : 0 = \sum_{ij} \left(\frac{d\mathcal{L}_M}{d \operatorname{Re} \partial_\alpha \psi^{ij}} \operatorname{Re} \left(\left[\psi^{\lozenge} \right] [\theta_a]^t \right)^i_j + \frac{d\mathcal{L}_M}{d \operatorname{Im} \partial_\alpha \psi^{ij}} \operatorname{Im} \left(\left[\psi^{\lozenge} \right] [\theta_a]^t \right)^i_j - \frac{d\mathcal{L}_M}{d \operatorname{Re} \grave{A}^a_\alpha} \right) \tag{33}$$

$$\forall a, \alpha : 0 = \sum_{ij} \left(\frac{d\mathcal{L}_M}{d \operatorname{Re} \partial_\alpha \psi^{ij}} \operatorname{Im} \left(\left[\psi^{\lozenge} \right] [\theta_a]^t \right)^i_j - \frac{d\mathcal{L}_M}{d \operatorname{Im} \partial_\alpha \psi^{ij}} \operatorname{Re} \left(\left[\psi^{\lozenge} \right] [\theta_a]^t \right)^i_j + \frac{d\mathcal{L}_M}{d \operatorname{Im} \grave{A}^a_\alpha} \right) \tag{34}$$

Terms in κ^a give :

$$\begin{gathered} \forall a : 0 = \sum_{ij} \tfrac{d\mathcal{L}_M}{d \operatorname{Re} \psi^{ij}} \operatorname{Re} \left([\kappa_a] [\psi] \right)^i_j + \tfrac{d\mathcal{L}_M}{d \operatorname{Im} \psi^{ij}} \operatorname{Im} \left([\kappa_a] [\psi] \right)^i_j \\ + \sum_{\alpha ij} \tfrac{d\mathcal{L}_M}{d \operatorname{Re} \partial_\alpha \psi^{ij}} \operatorname{Re} \left([\kappa_a] [\partial_\alpha \psi] \right)^i_j + \tfrac{d\mathcal{L}_M}{\partial \operatorname{Im} \partial_\alpha \psi^{ij}} \operatorname{Im} \left([\kappa_a] [\partial_\alpha \psi] \right)^i_j \\ + \sum_{b\alpha} \tfrac{d\mathcal{L}_M}{dG^b_\alpha} \left[\overrightarrow{\kappa}_a, G_\alpha \right]^b + \sum_{i\alpha} \tfrac{d\mathcal{L}_M}{dO'^i_\alpha} \left([\widetilde{\kappa}_a] [O'] \right)^i_\alpha + \tfrac{d\mathcal{L}_M}{d\partial_\beta O'^i_\alpha} \left([\widetilde{\kappa}_a] [O'] \right)^i_\alpha \end{gathered} \tag{35}$$

Terms in θ^a give :

$$\begin{gathered} \forall a : 0 = \sum_{ij} \tfrac{d\mathcal{L}_M}{d \operatorname{Re} \psi^{ij}} \operatorname{Re} \left([\psi] [\theta_a]^t \right)^i_j + \tfrac{d\mathcal{L}_M}{d \operatorname{Re} \partial_\alpha \psi^{ij}} \operatorname{Re} \left([\partial_\alpha \psi] [\theta_a]^t \right)^i_j \\ + \tfrac{d\mathcal{L}_M}{d \operatorname{Im} \psi^{ij}} \operatorname{Im} \left([\psi] [\theta_a]^t \right)^i_j + \tfrac{d\mathcal{L}_M}{d \operatorname{Im} \partial_\alpha \psi^{ij}} \operatorname{Im} \left([\partial_\alpha \psi] [\theta_a]^t \right)^i_j + \\ \sum_{b\alpha} \tfrac{d\mathcal{L}_M}{d \operatorname{Re} \grave{A}^b_\alpha} \operatorname{Re} \left[\overrightarrow{\theta}_a, \grave{A}_\alpha \right]^b + \tfrac{d\mathcal{L}_M}{d \operatorname{Im} \grave{A}^b_\alpha} \operatorname{Im} \left[\overrightarrow{\theta}_a, \grave{A}_\alpha \right]^b \end{gathered} \tag{36}$$

$$\forall a : 0 = \sum_{ij} - \tfrac{d\mathcal{L}_M}{d \operatorname{Re} \psi^{ij}} \operatorname{Im} \left([\psi] [\theta_a]^t \right)^i_j - \tfrac{\partial \mathcal{L}_M}{\partial \operatorname{Re} \partial_\alpha \psi^{ij}} \operatorname{Im} \left([\partial_\alpha \psi] [\theta_a]^t \right)^i_j \tag{37}$$

$+\frac{\partial \mathcal{L}_M}{\partial \operatorname{Im} \psi^{ij}} \operatorname{Re}\left([\psi]\left[\theta_a\right]^t\right)_j^i+\frac{\partial \mathcal{L}_M}{\partial \operatorname{Im} \partial_\alpha \psi^{ij}} \operatorname{Re}\left(\left[\partial_\alpha \psi\right]\left[\theta_a\right]^t\right)_j^i$
$+\sum_{b\alpha}-\frac{\partial \mathcal{L}_M}{\partial \operatorname{Re} \grave{A}_\alpha^b} \operatorname{Im}\left[\overrightarrow{\theta}_a, \grave{A}_\alpha\right]^b+\frac{\partial \mathcal{L}_M}{\partial \operatorname{Im} \grave{A}_\alpha^b} \operatorname{Re}\left[\overrightarrow{\theta}_a, \grave{A}_\alpha\right]^b$

2) By changing the variables :
$\partial_\alpha \psi^{ij} \rightarrow \nabla_\alpha \psi-\left(\left[\kappa_a\right]_k^i G_\alpha^a \psi^{kj}+\left[\theta_a\right]_k^j \grave{A}_\alpha^a \psi^{ik}\right)$
$\operatorname{Re} \partial_\alpha \psi^{ij} \rightarrow \operatorname{Re} \nabla_\alpha \psi-\left(G_\alpha^a \operatorname{Re}\left(\left[\kappa_a\right][\psi]\right)_j^i+\operatorname{Re} \grave{A}_\alpha^a \operatorname{Re}\left([\psi]\left[\theta_a\right]^t\right)_j^i-\operatorname{Im} \grave{A}_\alpha^a \operatorname{Im}\left([\psi]\left[\theta_a\right]^t\right)_j^i\right)$
$\operatorname{Im} \partial_\alpha \psi^{ij} \rightarrow \operatorname{Im} \nabla_\alpha \psi-\left(G_\alpha^a \operatorname{Im}\left(\left[\kappa_a\right][\psi]\right)_j^i+\operatorname{Re} \grave{A}_\alpha^a \operatorname{Im}\left([\psi]\left[\theta_a\right]^t\right)_j^i+\operatorname{Im} \grave{A}_\alpha^a \operatorname{Re}\left([\psi]\left[\theta_a\right]^t\right)_j^i\right)$
and expressing $\mathcal{L}_M$ as $\mathcal{L}\prime_M$ with these new arguments it comes :
$\frac{d \mathcal{L}_M}{d \operatorname{Re} \partial_\alpha \psi^{ij}}=\frac{d \mathcal{L}_M^{\prime}}{d \operatorname{Re} \nabla_\alpha \psi^{ij}} ; \frac{d \mathcal{L}_M}{d \operatorname{Im} \partial_\alpha \psi^{ij}}=\frac{d \mathcal{L}_M^{\prime}}{d \operatorname{Im} \nabla_\alpha \psi^{ij}}$
$\frac{d \mathcal{L}_M}{d G_\alpha^a}=\frac{\partial \mathcal{L}_M}{\partial G_\alpha^a}+\sum_{ij}\left(\frac{d \mathcal{L}_M}{d \operatorname{Re} \partial_\alpha \psi^{ij}} \operatorname{Re}\left(\left[\kappa_a\right]\left[\psi^{\Diamond}\right]\right)_j^i+\frac{d \mathcal{L}_M}{d \operatorname{Im} \partial_\alpha \psi^{ij}} \operatorname{Im}\left(\left[\kappa_a\right]\left[\psi^{\Diamond}\right]\right)_j^i\right)$
Equation 32 gives :

$$\forall a, \alpha: \frac{\partial \mathcal{L}_M}{\partial G_\alpha^a}=\sum_{i\beta} \frac{d \mathcal{L}_M}{d \partial_\alpha O_\beta^{\prime i}}\left(\left[\widetilde{\kappa}_a\right]\left[O^{\prime}\right]\right)_\beta^i \tag{38}$$

G factorizes through the covariant derivative if $\mathcal{L}_M$ does not depend on $\partial_\beta O_\alpha^{\prime i}$.

Equations 33 , 34 give :
$\frac{d \mathcal{L}_M}{d \operatorname{Re} \grave{A}_\alpha^a}=\sum_{ij} \frac{d \mathcal{L}_M}{d \operatorname{Re} \nabla_\alpha \psi^{ij}} \operatorname{Re}\left(\left[\psi^{\Diamond}\right]\left[\theta_a\right]^t\right)_j^i+\frac{d \mathcal{L}_M}{d \operatorname{Im} \nabla_\alpha \psi^{ij}} \operatorname{Im}\left(\left[\psi^{\Diamond}\right]\left[\theta_a\right]^t\right)_j^i$
$\frac{d \mathcal{L}_M}{d \operatorname{Im} \grave{A}_\alpha^a}=\sum_{ij}\left(-\frac{d \mathcal{L}_M}{d \operatorname{Re} \nabla_\alpha \psi^{ij}} \operatorname{Im}\left(\left[\psi^{\Diamond}\right]\left[\theta_a\right]^t\right)_j^i+\frac{d \mathcal{L}_M}{d \operatorname{Im} \nabla_\alpha \psi^{ij}} \operatorname{Re}\left(\left[\psi^{\Diamond}\right]\left[\theta_a\right]^t\right)_j^i\right)$

$$\frac{\partial \mathcal{L}_M}{\partial \operatorname{Re} \grave{A}_\alpha^a}=0 ; \frac{\partial \mathcal{L}_M}{\partial \operatorname{Im} \grave{A}_\alpha^a}=0 \tag{39}$$

À factorizes through the covariant derivative.

3) With :
$\frac{d \mathcal{L}_M}{d \operatorname{Re} \psi^{ij}}=\frac{\partial \mathcal{L}_M}{\partial \operatorname{Re} \psi^{ij}}+\sum_k \frac{d \mathcal{L}_M}{d \operatorname{Re} \nabla_\alpha \psi^{kj}} \operatorname{Re}\left[G_\alpha\right]_i^k+\frac{d \mathcal{L}_M}{d \operatorname{Im} \nabla_\alpha \psi^{kj}} \operatorname{Im}\left[G_\alpha\right]_i^k$
$+\frac{d \mathcal{L}_M}{d \operatorname{Re} \nabla_\alpha \psi^{ik}} \operatorname{Re}\left[\grave{A}_\alpha\right]_j^k+\frac{d \mathcal{L}_M}{d \operatorname{Im} \nabla_\alpha \psi^{ik}} \operatorname{Im}\left[\grave{A}_\alpha\right]_j^k$
$\frac{d \mathcal{L}_M}{d \operatorname{Im} \psi^{ij}}=\frac{\partial \mathcal{L}_M}{\partial \operatorname{Im} \psi^{ij}}+\sum_k-\frac{d \mathcal{L}_M}{d \operatorname{Re} \nabla_\alpha \psi^{kj}} \operatorname{Im}\left[G_\alpha\right]_i^k+\frac{d \mathcal{L}_M}{d \operatorname{Im} \nabla_\alpha \psi^{kj}} \operatorname{Re}\left[G_\alpha\right]_i^k$

$-\frac{d\mathcal{L}_M}{d\operatorname{Re}\nabla_\alpha\psi^{ik}}\operatorname{Im}\left[\grave{A}_\alpha\right]_j^k+\frac{d\mathcal{L}_M}{d\operatorname{Im}\nabla_\alpha\psi^{ik}}\operatorname{Re}\left[\grave{A}_\alpha\right]_j^k$

Thus the 3 equations left give :

$$\forall a:\sum_{ij}\frac{d\mathcal{L}_M}{d\operatorname{Re}\psi^{ij}}\operatorname{Re}\left(\left[\kappa_a\right]\left[\psi\right]\right)_j^i+\frac{d\mathcal{L}_M}{d\operatorname{Im}\psi^{ij}}\operatorname{Im}\left(\left[\kappa_a\right]\left[\psi\right]\right)_j^i$$
$$+\sum_{\alpha ij}\frac{d\mathcal{L}_M}{d\operatorname{Re}\partial_\alpha\psi^{ij}}\operatorname{Re}\left(\left[\kappa_a\right]\left[\partial_\alpha\psi\right]\right)_j^i+\frac{d\mathcal{L}_M}{\partial\operatorname{Im}\partial_\alpha\psi^{ij}}\operatorname{Im}\left(\left[\kappa_a\right]\left[\partial_\alpha\psi\right]\right)_j^i$$
$$+\sum_{b\alpha}\frac{d\mathcal{L}_M}{dB_\alpha^b}\left[\overrightarrow{\kappa}_a,G_\alpha\right]^b+\sum_{i\alpha}\frac{d\mathcal{L}_M}{dO_\alpha'^i}\left(\left[\widetilde{\kappa}_a\right]\left[O'\right]\right)_\alpha^i+\frac{d\mathcal{L}_M}{d\partial_\beta O_\alpha'^i}\left(\left[\widetilde{\kappa}_a\right]\left[O'\right]\right)_\alpha^i=0 \tag{40}$$

$$\forall a:0=\sum_{ij\alpha}\frac{\partial L_M}{\partial\operatorname{Re}\psi^{ij}}\operatorname{Re}\left(\left[\kappa_a\right]\left[\psi\right]\right)_j^i+\frac{\partial L_M}{\partial\operatorname{Im}\psi^{ij}}\operatorname{Im}\left(\left[\kappa_a\right]\left[\psi\right]\right)_j^i$$
$$+\frac{dL_M}{d\operatorname{Re}\nabla_\alpha\psi^{ij}}\operatorname{Re}\left(\left[\kappa_a\right]\left[\nabla_\alpha\psi\right]\right)_j^i+\frac{dL_M}{d\operatorname{Im}\nabla_\alpha\psi^{ij}}\operatorname{Im}\left(\left[\kappa_a\right]\left[\nabla_\alpha\psi\right]\right)_j^i$$
$$+\sum_{\alpha b}\frac{\partial L_M}{\partial G_\alpha^b}\left[\overrightarrow{\kappa}_a,G_\alpha\right]^b+\sum_{ij}\left[\widetilde{\kappa}_a\right]_j^i\sum_{i\alpha\beta}\frac{dL_M}{dO_\alpha'^i}O_\alpha'^j+\frac{dL_M}{d\partial_\beta O_\alpha'^i}\partial_\beta O_\alpha'^j \tag{41}$$

$$\forall a:0=\sum_{ij\alpha}\frac{\partial L_M}{\partial\operatorname{Re}\psi^{ij}}\operatorname{Re}\left(\left[\psi\right]\left[\theta_a\right]^t\right)_j^i+\frac{\partial L_M}{\partial\operatorname{Im}\psi^{ij}}\operatorname{Im}\left(\left[\psi\right]\left[\theta_a\right]^t\right)_j^i$$
$$+\frac{dL_M}{d\operatorname{Re}\nabla_\alpha\psi^{ij}}\operatorname{Re}\left(\left[\nabla_\alpha\psi\right]\left[\theta_a\right]^t\right)_j^i+\frac{dL_M}{d\operatorname{Im}\nabla_\alpha\psi^{ij}}\operatorname{Im}\left(\left[\nabla_\alpha\psi\right]\left[\theta_a\right]^t\right)_j^i \tag{42}$$

$$\forall a:0=\sum_{ij\alpha}-\frac{\partial L_M}{\partial\operatorname{Re}\psi^{ij}}\operatorname{Im}\left(\left[\psi\right]\left[\theta_a\right]^t\right)_j^i+\frac{\partial L_M}{\partial\operatorname{Im}\psi^{ij}}\operatorname{Re}\left(\left[\psi\right]\left[\theta_a\right]^t\right)_j^i$$
$$-\frac{dL_M}{d\operatorname{Re}\nabla_\alpha\psi^{ij}}\operatorname{Im}\left(\left[\nabla_\alpha\psi\right]\left[\theta_a\right]^t\right)_j^i+\frac{dL_M}{d\operatorname{Im}\nabla_\alpha\psi^{ij}}\operatorname{Re}\left(\left[\nabla_\alpha\psi\right]\left[\theta_a\right]^t\right)_j^i$$

6 COVARIANCE

Covariance needs that the lagrangian be invariant under a change of the map of the underlying manifold. The map $\widetilde{f}:\Omega\rightarrow\Omega$ is not affected in such an operation.

One can proceed as in the previous section, but the group involved in covariance is the group of general diffeomorphisms on M, it would be quite restrictive to reduce it to some one parameter group. Moreover the traditional way is simple and gives some results which will prove very useful. So we follow the general method as presented in Lovelock [18].

1) A change of chart is a coordinates transformation $\xi \to \widehat{\xi} = F(\xi)$ characterized by the jacobian $J^\alpha_\beta(m) = \frac{\partial_\alpha}{\widehat{\partial}_\beta} = [F'(m)]^\alpha_\beta$ whose matrix is in GL(4). We denote its inverse matrix $K = J^{-1}$. It induces the following transformations :

$\partial_\alpha \to \widehat{\partial}_\alpha = J^\beta_\alpha \partial_\beta$

$dx^\alpha \to \widehat{dx}^\alpha = K^\alpha_\beta dx^\beta$

$\varpi_0 \to \widehat{\varpi}_0 = (\det K)\,\varpi_0$

on vector fields on M : $X^\alpha \partial_\alpha = \widehat{X}^\alpha \widehat{\partial}_\alpha \Rightarrow \widehat{X}^\alpha = K^\alpha_\beta X^\beta$

on forms on M : $X_\alpha \partial^\alpha = \widehat{X}_\alpha \widehat{\partial}^\alpha \Rightarrow \widehat{X}_\alpha = J^\beta_\alpha X_\beta$

J^1Z is identical to $JZ \otimes TV^* = L(TV; JZ)$ so the transformations on J^1Z are deduced from the transformations on TM.

ψ is unchanged, all the other quantities are vectors and 1 or 2 forms :

$V^\alpha \to \widehat{V}^\alpha = K^\alpha_\beta V^\beta$

$O'^i_\alpha \to \widehat{O}'^i_\alpha = O'^i_\lambda J^\lambda_\alpha \Rightarrow \det O' \to \det \widehat{O}' = \det J \det O'$

$\partial_\alpha \psi^{ij} \to \widehat{\partial_\alpha \psi^{ij}} = J^\lambda_\alpha \partial_\lambda \psi^{ij}$

$\grave{A}^a_\alpha \to \widehat{\grave{A}}^a_\alpha = \grave{A}^a_\lambda J^\lambda_\alpha$

$G^a_\alpha \to \widehat{G}^a_\alpha = J^\lambda_\alpha G^a_\lambda$

$\partial_\alpha G^a_\beta \to \widehat{\partial_\alpha G^a_\beta} = J^\lambda_\alpha J^\mu_\beta \partial_\lambda G^a_\mu$

$\partial_\alpha \grave{A}^a_\beta \to \widehat{\partial_\alpha \grave{A}^a_\beta} = J^\lambda_\alpha J^\mu_\beta \partial_\lambda \grave{A}^a_\mu$

$\partial_\beta O'^i_\alpha \to \widehat{\partial_\beta O'}^i_\alpha = J^\lambda_\alpha J^\mu_\beta \left(\partial_\lambda O'^i_\mu\right)$

and also :

$\nabla_\alpha \psi^{ij} \to \widehat{\nabla_\alpha \psi^{ij}} = J^\lambda_\alpha \nabla_\lambda \psi^{ij}$

$\mathcal{F}^a_{A,\alpha\beta} \to \widehat{\mathcal{F}^a_{A,\alpha\beta}} = J^\lambda_\alpha J^\mu_\beta \mathcal{F}^a_{A,\lambda\mu}$

$\mathcal{F}^a_{G,\alpha\beta} \to \widehat{\mathcal{F}^a_{G,\alpha\beta}} = J^\lambda_\alpha J^\mu_\beta \mathcal{F}^a_{G,\lambda\mu}$

$\mathcal{F}^a_{A,\alpha\beta} \to \widehat{\mathcal{F}^a_{A,\alpha\beta}} = J^\lambda_\alpha J^\mu_\beta \mathcal{F}^a_{A,\lambda\mu}$

J and K being real matrix all the formula stand for real and imaginary quantities.

$\widetilde{f} : \Omega \to \Omega$ is unchanged and the variables $\left(z^{i\Diamond}, z^{i\Diamond}_\alpha\right) = \widetilde{f}^*\left(z^i, z^i_\alpha\right)$ transform as $\left(z^i, z^i_\alpha\right)$.

2) We have for the 4-form on M :

$(NL_M + L_F)\,\varpi_4 = (NL_M + L_F)\det O'\varpi_0$

$\to \left(N\widehat{L}_M + \widehat{L}_F\right)\det \widehat{O}'\widehat{\varpi}_0$

$= \left(N\widehat{L}_M + \widehat{L}_F\right) \det J \det K \det O' \varpi_0$
$= \left(N\widehat{L}_M + \widehat{L}_F\right) \det O' \varpi_0$
$\Rightarrow (NL_M + L_F) = \left(N\widehat{L}_M + \widehat{L}_F\right)$
$\widetilde{f}$ acts only on the first part, so we must have the two identities :
$L_M\left(z^i, z^i_\alpha\right) = \widehat{L}_M\left(\widehat{z}^i, \widehat{z}^i_\alpha\right)$
$L_F\left(z^i, z^i_\alpha\right) = \widehat{L}_F\left(\widehat{z}^i, \widehat{z}^i_\alpha\right)$
$L_M\left(V^\alpha, \operatorname{Re}\psi^{ij}, \operatorname{Im}\psi^{ij}, \operatorname{Re}\partial_\alpha\psi^{ij}, \operatorname{Im}\partial_\alpha\psi^{ij}, G^a_\alpha, \operatorname{Re}\grave{A}^a_\alpha, \operatorname{Im}\grave{A}^a_\alpha, O^i_\alpha, \partial_\beta O'^i_\alpha\right)$
$= \widehat{L}_M(K^\alpha_\beta V^\beta, \operatorname{Re}\psi^{ij}, \operatorname{Im}\psi^{ij}, J^\lambda_\alpha \operatorname{Re}\partial_\lambda\psi^{ij}, J^\lambda_\alpha \operatorname{Im}\partial_\lambda\psi^{ij}, J^\lambda_\alpha G^a_\lambda,$
$\quad J^\lambda_\alpha \operatorname{Re}\grave{A}^a_\lambda, J^\lambda_\alpha \operatorname{Im}\grave{A}^a_\lambda, O'^i_\lambda J^\lambda_\alpha, J^\lambda_\alpha J^\mu_\beta \partial_\lambda O'^i_\mu)$
$L_F\left(G^a_\alpha, \partial_\alpha G^a_\beta, \operatorname{Re}\grave{A}^a_\alpha, \operatorname{Im}\grave{A}^a_\alpha, \operatorname{Re}\partial_\alpha\grave{A}^a_\beta, \operatorname{Im}\partial_\alpha\grave{A}^a_\beta, O'^i_i, \partial_\beta O'^i_\alpha\right)$
$= \widehat{L}_F(J^\lambda_\alpha G^a_\lambda, J^\lambda_\alpha J^\mu_\beta \partial_\lambda G^a_\mu, J^\lambda_\alpha \operatorname{Re}\grave{A}^a_\lambda, J^\lambda_\alpha \operatorname{Im}\grave{A}^a_\lambda, J^\lambda_\alpha J^\mu_\beta \operatorname{Re}\partial_\lambda\grave{A}^a_\mu,$
$\quad J^\lambda_\alpha J^\mu_\beta \operatorname{Im}\partial_\lambda\grave{A}^a_\mu, , O'^i_\lambda J^\lambda_\alpha, J^\lambda_\alpha J^\mu_\beta \partial_\lambda O'^i_\mu)$

3) By differentiating with respect to J^β_α one gets the identities :
with $\frac{d}{dJ^\beta_\alpha} = \sum_{\lambda\mu} \frac{d}{dK^\lambda_\mu}\frac{dK^\lambda_\mu}{dJ^\beta_\alpha} = \sum_{\lambda\mu}\left(-K^\lambda_\beta K^\alpha_\mu\right)\frac{d}{dK^\lambda_\mu}$
$\forall\alpha,\beta : 0 = \sum_{\lambda\mu}\{\sum_\gamma \frac{d\widehat{L}_M}{d\widehat{V}^\gamma}\left(-K^\lambda_\beta K^\alpha_\mu\right)\frac{d\widehat{V}^\gamma}{dK^\lambda_\mu}$
$+ \sum_{i,j} \frac{d\widehat{L}_M}{d\operatorname{Re}\widehat{\partial_\lambda\psi^{ij}}}\frac{\partial J^\mu_\lambda \operatorname{Re}\partial_\mu\psi^{ij}}{\partial J^\beta_\alpha} + \frac{d\widehat{L}_M}{d\operatorname{Im}\widehat{\partial_\lambda\psi^{ij}}}\frac{\partial J^\mu_\lambda \operatorname{Im}\partial_\mu\psi^{ij}}{\partial J^\beta_\alpha}$
$+ \sum_a \frac{d\widehat{L}_M}{d\widehat{G^a_\lambda}}\frac{\partial J^\mu_\lambda G^a_\mu}{\partial J^\beta_\alpha} + \frac{d\widehat{L}_M}{d\operatorname{Re}\widehat{\grave{A}^a_\lambda}}\frac{\partial J^\mu_\lambda \operatorname{Re}\grave{A}^a_\mu}{\partial J^\beta_\alpha} + \frac{d\widehat{L}_M}{d\operatorname{Im}\widehat{\grave{A}^a_\lambda}}\frac{\partial J^\mu_\lambda \operatorname{Im}\grave{A}^a_\mu}{\partial J^\beta_\alpha}$
$+ \sum_i \frac{d\widehat{L}_M}{d\widehat{O'^i_\lambda}}\frac{\partial J^\mu_\lambda O'^i_\mu}{\partial J^\beta_\alpha} + \sum_{\xi\eta}\frac{d\widehat{L}_M}{d\widehat{\partial_\lambda O^{i\prime}_\mu}}\frac{\partial J^\xi_\lambda J^\eta_\mu \partial_\xi O'^i_\eta}{\partial J^\beta_\alpha}\}$
$\forall\alpha,\beta : 0 = \sum_\gamma \frac{d\widehat{L}_M}{d\widehat{V}^\gamma}\left(-K^\gamma_\beta K^\alpha_\mu V^\mu\right) + \sum_{i,j}\frac{d\widehat{L}_M}{d\operatorname{Re}\widehat{\partial_\alpha\psi^{ij}}}\operatorname{Re}\partial_\beta\psi^{ij} + \frac{d\widehat{L}_M}{d\operatorname{Im}\widehat{\partial_\alpha\psi^{ij}}}\operatorname{Im}\partial_\beta\psi^{ij}$
$+ \sum_a \frac{d\widehat{L}_M}{d\widehat{G^a_\alpha}}G^a_\beta + \frac{d\widehat{L}_M}{d\operatorname{Re}\widehat{\grave{A}^a_\alpha}}\operatorname{Re}\grave{A}^a_\beta + \frac{d\widehat{L}_M}{d\operatorname{Im}\widehat{\grave{A}^a_\alpha}}\operatorname{Im}\grave{A}^a_\beta + \sum_i \frac{d\widehat{L}_M}{d\widehat{O'^i_\alpha}}O'^i_\beta$
$+ \sum_{i,\lambda,\mu}\frac{d\widehat{L}_M}{d\widehat{\partial_\alpha O^{i\prime}_\lambda}}J^\mu_\lambda \partial_\beta O'^i_\mu + \frac{d\widehat{L}_M}{d\widehat{\partial_\lambda O^{i\prime}_\alpha}}J^\mu_\lambda \partial_\mu O'^i_\beta$
and
$\forall\alpha,\beta : 0 = \sum_{\lambda\mu}\{\sum_a \frac{d\widehat{L}_F}{d\widehat{G^a_\lambda}}\frac{\partial J^\mu_\lambda G^a_\mu}{\partial J^\beta_\alpha} + \frac{d\widehat{L}_F}{d\operatorname{Re}\widehat{\grave{A}^a_\lambda}}\frac{\partial J^\mu_\lambda \operatorname{Re}\grave{A}^a_\mu}{\partial J^\beta_\alpha} + \frac{d\widehat{L}_F}{d\operatorname{Im}\widehat{\grave{A}^a_\lambda}}\frac{\partial J^\mu_\lambda \operatorname{Im}\grave{A}^a_\mu}{\partial J^\beta_\alpha}$
$+ \sum_{\xi\eta}\left(\frac{d\widehat{L}_F}{d\operatorname{Re}\widehat{\partial_\lambda\grave{A}^a_\mu}}\frac{\partial J^\xi_\lambda J^\eta_\mu \partial_\xi \operatorname{Re}\grave{A}^a_\eta}{\partial J^\beta_\alpha} + \frac{d\widehat{L}_F}{d\operatorname{Im}\widehat{\partial_\lambda\grave{A}^a_\mu}}\frac{\partial J^\xi_\lambda J^\eta_\mu \partial_\xi \operatorname{Im}\grave{A}^a_\eta}{\partial J^\beta_\alpha} + \frac{d\widehat{L}_F}{d\widehat{\partial_\lambda G^a_\mu}}\frac{\partial J^\xi_\lambda J^\eta_\mu \partial_\xi G^a_\eta}{\partial J^\beta_\alpha}\right)$
$+ \sum_i \frac{d\widehat{L}_F}{d\widehat{O'^i_\lambda}}\frac{\partial J^\mu_\lambda O'^i_\mu}{\partial J^\beta_\alpha} + \sum_{\xi\eta}\frac{d\widehat{L}_F}{d\widehat{\partial_\lambda O^{i\prime}_\mu}}\frac{\partial J^\xi_\lambda J^\eta_\mu \partial_\xi O'^i_\eta}{\partial J^\beta_\alpha}\}$

$\forall\alpha,\beta : 0 = \sum_a \frac{d\widehat{L}_F}{d\widehat{G^a_\alpha}} G^a_\beta + \frac{d\widehat{L}_F}{d\operatorname{Re}\widehat{\grave{A}^a_\alpha}} \operatorname{Re}\grave{A}^a_\beta + \frac{d\widehat{L}_F}{d\operatorname{Im}\widehat{\grave{A}^a_\alpha}} \operatorname{Im}\grave{A}^a_\beta$
$+ \sum_a \frac{d\widehat{L}_F}{d\widehat{O'^i_\alpha}} O'^i_\beta + \sum_{i,\lambda,\mu} \frac{d\widehat{L}_F}{d\widehat{\partial_\alpha O'^i_\lambda}} J^\mu_\lambda \partial_\beta O'^i_\mu + \frac{d\widehat{L}_F}{d\widehat{\partial_\lambda O'^i_\alpha}} J^\mu_\lambda \partial_\mu O'^i_\beta$
$+ \sum_{\lambda\mu} (\frac{d\widehat{L}_F}{d\operatorname{Re}\widehat{\partial_\alpha \grave{A}^a_\lambda}} J^\mu_\lambda \operatorname{Re}\left(\partial_\beta \grave{A}^a_\mu - \partial_\mu \grave{A}^a_\beta\right)$
$+ \frac{d\widehat{L}_F}{d\operatorname{Im}\widehat{\partial_\alpha \grave{A}^a_\lambda}} J^\mu_\lambda \operatorname{Im}\left(\partial_\beta \grave{A}^a_\mu - \partial_\mu \grave{A}^a_\beta\right) + \frac{d\widehat{L}_F}{d\widehat{\partial_\alpha G^a_\lambda}} J^\mu_\lambda \left(\partial_\beta G^a_\mu - \partial_\mu G^a_\beta\right))$

4) By differentiating with respect to the original arguments one gets
$\frac{\partial L_M}{\partial z^i} = \frac{\partial \widehat{L}_M}{\partial \widehat{z}^i} \frac{\partial \widehat{z}^i}{\partial z^i}$ which reads :
$\frac{dL_M}{dV^\alpha} = \sum_\beta \frac{d\widehat{L}_M}{d\widehat{V}^\beta} \frac{d\widehat{V}^\beta}{dV^\alpha} = \sum_\beta \frac{d\widehat{L}_M}{d\widehat{V}^\beta} K^\beta_\alpha \Leftrightarrow \frac{d\widehat{L}_M}{d\widehat{V}^\alpha} = \sum_\beta \frac{dL_M}{dV^\beta} J^\beta_\alpha$
$\frac{dL_M}{d\operatorname{Re}\psi^{ij}} = \frac{d\widehat{L}_M}{d\widehat{\operatorname{Re}\psi}^{ij}}; \frac{dL_M}{d\operatorname{Im}\psi^{ij}} = \frac{d\widehat{L}_M}{d\widehat{\operatorname{Im}\psi}^{ij}}$
$\frac{dL_M}{d\operatorname{Re}\partial_\alpha\psi^{ij}} = \sum_\lambda J^\alpha_\lambda \frac{d\widehat{L}_M}{d\operatorname{Re}\widehat{\partial_\lambda\psi}^{ij}}; \frac{dL_M}{d\operatorname{Im}\partial_\alpha\psi^{ij}} = \sum_\lambda J^\alpha_\lambda \frac{d\widehat{L}_M}{d\operatorname{Im}\widehat{\partial_\lambda\psi}^{ij}}$
$\frac{dL_M}{\partial G^a_\alpha} = J^\alpha_\lambda \frac{d\widehat{L}_M}{d\widehat{G}^a_\lambda}; \frac{dL_F}{\partial G^a_\alpha} = J^\alpha_\lambda \frac{d\widehat{L}_F}{d\widehat{G}^a_\lambda}$
$\frac{dL_M}{dO'^i_\alpha} = J^\alpha_\lambda \frac{d\widehat{L}_M}{d\widehat{O}'^i_\lambda}; \frac{dL_F}{dO'^i_\alpha} = J^\alpha_\lambda \frac{d\widehat{L}_F}{d\widehat{O}'^i_\lambda}$
$\frac{dL_M}{d\partial_\beta O'^i_\alpha} = J^\alpha_\lambda J^\beta_\mu \frac{d\widehat{L}_M}{d\partial_\mu \widehat{O}'^i_\lambda}; \frac{dL_F}{d\partial_\beta O'^i_\alpha} = J^\alpha_\lambda J^\beta_\mu \frac{d\widehat{L}_F}{d\partial_\mu \widehat{O}'^i_\lambda}$
$\frac{dL_F}{d\operatorname{Re}\partial_\alpha \grave{A}^a_\beta} = J^\alpha_\lambda J^\beta_\mu \frac{d\widehat{L}_F}{d\operatorname{Re}\widehat{\partial_\lambda \grave{A}^a_\mu}}; \frac{dL_F}{d\operatorname{Im}\partial_\alpha \grave{A}^a_\beta} = J^\alpha_\lambda J^\beta_\mu \frac{d\widehat{L}_F}{d\operatorname{Im}\widehat{\partial_\lambda \grave{A}^a_\mu}};$

Some of these partial derivatives transform as composants of tensors, and therefore we can introduce the corresponding tensorial objects :
$\frac{dL_M}{d\operatorname{Re}\psi^{ij}}, \frac{dL_M}{d\operatorname{Im}\psi^{ij}}$ are functions over M
$\frac{dL_M}{dV^\alpha}$ are components of a one form field: $\sum_\alpha \frac{dL_M}{dV^\alpha} dx^\alpha$
$\frac{dL_M}{d\operatorname{Re}\partial_\alpha\psi^{ij}}, \frac{dL_M}{d\operatorname{Im}\partial_\alpha\psi^{ij}}, \frac{dL_M}{dG^a_\alpha}, \frac{dL_F}{dG^a_\alpha}, \frac{dL_M}{dO'^i_\alpha}, \frac{dL_F}{dO'^i_\alpha}$ are components of vector fields :
$\sum_\alpha \frac{dL_M}{d\operatorname{Re}\partial_\alpha\psi^{ij}} \partial_\alpha, \sum_\alpha \frac{dL_M}{d\operatorname{Im}\partial_\alpha\psi^{ij}} \partial_\alpha, \sum_\alpha \frac{dL_M}{dG^a_\alpha} \partial_\alpha, \sum_\alpha \frac{dL_F}{dG^a_\alpha} \partial_\alpha, \sum_\alpha \frac{dL_M}{dO'^i_\alpha} \partial_\alpha$
$\frac{dL_F}{d\operatorname{Re}\partial_\alpha \grave{A}^a_\beta}, \frac{dL_F}{d\operatorname{Im}\partial_\alpha \grave{A}^a_\beta}, \frac{dL_F}{d\partial_\alpha G^a_\beta}$ are components of anti-symmetric bi-vector fields :
$\sum_{\{\alpha\beta\}} \frac{dL_F}{d\operatorname{Re}\partial_\alpha \grave{A}^a_\beta} \partial_\alpha \wedge \partial_\beta, \sum_{\{\alpha\beta\}} \frac{dL_F}{d\operatorname{Im}\partial_\alpha \grave{A}^a_\beta} \partial_\alpha \wedge \partial_\beta, \sum_{\{\alpha\beta\}} \frac{dL_F}{d\partial_\alpha G^a_\beta} \partial_\alpha \wedge \partial_\beta,$
$\frac{dL_M}{d\partial_\beta O'^i_\alpha}, \frac{dL_F}{d\partial_\beta O'^i_\alpha}$ are components of bi-tensor fields : $\sum_{\alpha\beta} \frac{dL_M}{d\partial_\alpha O'^i_\beta} \partial_\alpha \otimes \partial_\beta, \sum_{\alpha\beta} \frac{dL_F}{d\partial_\alpha O'^i_\beta} \partial_\alpha \otimes \partial_\beta$

But the quantities such as $\frac{\partial L_F \det O'}{\partial \operatorname{Re}\mathcal{F}^a_{A,\alpha\beta}} = \frac{\partial L_F}{\partial \operatorname{Re}\mathcal{F}^a_{A,\alpha\beta}} \det O'$ are **not** tensorial.

5) By putting $J^\alpha_\beta = \delta^\alpha_\beta$ we see that the values of the partial derivatives are unchanged. So the two previous identities give :

$$\begin{aligned}&\forall\alpha,\beta: 0 = -\frac{dL_M}{dV^\beta}V^\alpha + \sum_{i,j}\frac{dL_M}{d\operatorname{Re}\partial_\alpha\psi^{ij}}\operatorname{Re}\partial_\beta\psi^{ij} + \frac{dL_M}{d\operatorname{Im}\partial_\alpha\psi^{ij}}\operatorname{Im}\partial_\beta\psi^{ij}\\ &+\sum_a \frac{dL_M}{dG^a_\alpha}G^a_\beta + \frac{dL_M}{d\operatorname{Re}\grave{A}^a_\alpha}\operatorname{Re}\grave{A}^a_\beta + \frac{dL_M}{d\operatorname{Im}\grave{A}^a_\alpha}\operatorname{Im}\grave{A}^a_\beta\\ &+\sum_i\left(\frac{dL_M}{dO'^i_\alpha}O'^i_\beta + \sum_\lambda \frac{dL_M}{d\partial_\lambda O'^i_\alpha}\partial_\lambda O'^i_\beta + \frac{dL_M}{d\partial_\alpha O'^i_\lambda}\partial_\beta O'^i_\lambda\right)\end{aligned} \tag{43}$$

$$\begin{aligned}&\forall\alpha,\beta: 0 = \sum_a \frac{dL_F}{dG^a_\alpha}G^a_\beta + \frac{dL_F}{d\operatorname{Re}\grave{A}^a_\alpha}\operatorname{Re}\grave{A}^a_\beta + \frac{dL_F}{d\operatorname{Im}\grave{A}^a_\alpha}\operatorname{Im}\grave{A}^a_\beta\\ &+\sum_{a,\lambda}(\frac{dL_F}{d\operatorname{Re}\partial_\alpha\grave{A}^a_\lambda}\operatorname{Re}\left(\partial_\beta\grave{A}^a_\lambda - \partial_\lambda\grave{A}^a_\beta\right) + \frac{dL_F}{d\operatorname{Im}\partial_\alpha\grave{A}^a_\lambda}\operatorname{Im}\left(\partial_\beta\grave{A}^a_\lambda - \partial_\lambda\grave{A}^a_\beta\right)\\ &+\frac{dL_F}{d\partial_\alpha G^a_\lambda}\left(\partial_\beta G^a_\lambda - \partial_\lambda G^a_\beta\right))\\ &+\sum_i\left(\frac{dL_F}{dO'^i_\alpha}O'^i_\beta + \sum_\lambda \frac{dL_F}{d\partial_\lambda O'^i_\alpha}\partial_\lambda O'^i_\beta + \frac{dL_F}{d\partial_\alpha O'^i_\lambda}\partial_\beta O'^i_\lambda\right)\end{aligned} \tag{44}$$

6) By proceeding to the same calculations with the lagrangians :
$L_M\left(\operatorname{Re}\psi^{ij}, \operatorname{Im}\psi^{ij}, \operatorname{Re}\nabla_\alpha\psi^{ij}, \operatorname{Im}\nabla_\alpha\psi^{ij}, G^a_\alpha, O^i_\alpha, \partial_\beta O'^i_\alpha\right)$,
$L_F\left(\operatorname{Re}\mathcal{F}^a_{A,\alpha\beta}, \operatorname{Im}\mathcal{F}^a_{A,\alpha\beta}, \mathcal{F}^a_{G,\alpha\beta}, O'^i_i, \partial_\beta O'^i_\alpha\right)$
one can check that the following quantities are tensorial :

$\frac{\partial L_M}{\partial\operatorname{Re}\nabla_\alpha\psi^{ij}}, \frac{\partial L_M}{\partial\operatorname{Im}\nabla_\alpha\psi^{ij}}$ are components of vector fields : $\sum_\alpha \frac{\partial L_M}{\partial\operatorname{Re}\nabla_\alpha\psi^{ij}}\partial_\alpha, \sum_\alpha \frac{\partial L_M}{\partial\operatorname{Im}\nabla_\alpha\psi^{ij}}\partial_\alpha$
$\frac{\partial L_F}{\partial\operatorname{Re}\mathcal{F}^a_{A,\alpha\beta}}, \frac{\partial L_F}{\partial\operatorname{Im}\mathcal{F}^a_{A,\alpha\beta}}, \frac{\partial L_F}{\partial\mathcal{F}^a_{G,\alpha\beta}}$ are components of anti-symmetric bi-vector fields
:
$\sum_{\alpha\beta}\frac{\partial L_F}{\partial\operatorname{Re}\mathcal{F}^a_{A,\alpha\beta}}\partial_\alpha\wedge\partial_\beta, \sum_{\alpha\beta}\frac{\partial L_F}{\partial\operatorname{Im}\mathcal{F}^a_{A,\alpha\beta}}\partial_\alpha\wedge\partial_\beta, \sum_{\alpha\beta}\frac{\partial L_F}{\partial\mathcal{F}^a_{G,\alpha\beta}}\partial_\alpha\wedge\partial_\beta$

7) With the identities from the previous section the two equations 43,44 become :

$$\forall\alpha,\beta: 0 = -\frac{dL_M}{dV^\beta}V^\alpha + \sum_{i,j}\left(\frac{dL_M}{d\operatorname{Re}\nabla_\alpha\psi^{ij}}\operatorname{Re}\nabla_\beta\psi^{ij} + \frac{dL_M}{d\operatorname{Im}\nabla_\alpha\psi^{ij}}\operatorname{Im}\nabla_\beta\psi^{ij}\right) \tag{45}$$

$$+ \sum_a \frac{\partial L_M}{\partial G^a_\alpha} G^a_\beta + \sum_i \frac{dL_M}{dO'^i_\alpha} O'^i_\beta + \sum_{i\lambda} \frac{dL_M}{d\partial_\lambda O'^i_\alpha} \left(\partial_\lambda O'^i_\beta\right) + \frac{dL_M}{d\partial_\alpha O'^i_\lambda} \left(\partial_\beta O'^i_\lambda\right) \tag{46}$$

$$\forall \alpha, \beta : 0 = 2 \sum_{a\lambda} \left(\frac{dL_F}{d \operatorname{Re} \mathcal{F}^a_{A,\alpha\lambda}} \operatorname{Re} \mathcal{F}^a_{A,\beta\lambda} + \frac{dL_F}{d \operatorname{Im} \mathcal{F}^a_{A,\alpha\lambda}} \operatorname{Im} \mathcal{F}^a_{A,\beta\lambda} + \frac{dL_F}{d\mathcal{F}^a_{G,\alpha\lambda}} \mathcal{F}^a_{G,\beta\lambda} \right)$$
$$+ \sum_a \frac{\partial L_F}{\partial G^a_\alpha} G^a_\beta + \sum_i \frac{dL_F}{dO'^i_\alpha} O'^i_\beta + \sum_{i\lambda} \frac{dL_F}{d\partial_\lambda O^{i\prime}_\alpha} \left(\partial_\lambda O'^i_\beta\right) + \frac{dL_F}{d\partial_\alpha O^{i\prime}_\lambda} \left(\partial_\beta O'^i_\lambda\right)$$

Part III
LAGRANGE EQUATIONS

7 PRINCIPLES

Functions for which the action is stationary are given by the standard variational calculus, in the form of the Euler-Lagrange equations. We will review them below. But our problem is more complicated due to the $\widetilde{f}$ map, which needs the use of functional derivatives techniques.

7.1 Variational calculus

1) We have seen previously how a projectable vector field on JZ is the generator of a one parameter group Φ_τ^Y which can be extended to J^1Z . The group $\Phi_\tau^{J^1Y}$ induces a deformation of a section j^1Z on J^1Z : $j^1Z \to \Phi_\tau^{J^1Y}(j^1Z)$ and of the value of the lagrangian and action :

$j^1Z^*\mathcal{L}\varpi_0 \to \left(\Phi_\tau^{J^1Y}(j^1Z)\right)^*\mathcal{L}\varpi_0$

$S(Z) = \int_\Omega (j^1Z)^* \mathcal{L}\varpi_0 \to S(\tau, Y) = \int_\Omega \left(\Phi_\tau^{J^1Y}(j^1Z)\right)^*\mathcal{L}\varpi_0$

For Y fixed $S(\tau, Y)$ is a function of the scalar τ. By derivation with respect to τ in $\tau = 0$ one gets the variational derivative of $\mathcal{L}$ along Y :

$\frac{d}{d\tau}S(\tau, Y)|_{\tau=0} = \int_\Omega \frac{d}{d\tau}\left(\left(\Phi_\tau^{J^1Y}(j^1Z)\right)^*\mathcal{L}\varpi_0\right)|_{\tau=0} = \int_\Omega (j^1Z)^* \pounds_{j^1Y}\mathcal{L}\varpi_0$

where $\pounds_{j^1Y}\mathcal{L}\varpi_0$ is the Lie derivative of $\mathcal{L}\varpi_0$ along the field J^1Y.

The solutions of the variational problem are taken as the sections Z such that S is stationary for any projectable vector field Y with support included in Ω, that is : $\frac{d}{d\tau}S(\tau, Y)|_{\tau=0} = 0$ or $\int_\Omega (j^1Z)^* \pounds_{j^1Y}\mathcal{L}\varpi_0 = 0$.

2) The first variation formula of variational calculus gives the value of this Lie derivative (Giachetta [5] p.75, Krupka [15]) :

$$\pounds_{j^1Y}\mathcal{L}\varpi_0 = \sum_i Y^i E_i \varpi_0 + hd\left(i_{j^1Y}\Theta_L\right) \tag{47}$$

where

$\Theta_L = \mathcal{L}\varpi_0 + \frac{\partial \mathcal{L}}{\partial z_\alpha^i}\varpi^i \wedge i_{\partial_\alpha}\varpi_0$ is the Poincaré-Cartan Lepage equivalent of the lagrangian, with $\varpi^i = \left(dz^i - z_\alpha^i d\xi^\alpha\right)$

$E_i = \frac{\partial \mathcal{L}}{\partial z^i} - \sum_{\alpha=1}^n \frac{d}{d\xi^\alpha}\left(\frac{\partial \mathcal{L}}{\partial z^i_\alpha}\right)$; $E = E_i \varpi^i \wedge \varpi_0$ is the Euler-Lagrange form

h is the horizontalization, an exterior product preserving morphism :

$h : \Lambda_q(J^r Z; R) \to \Lambda_q^H(J^{r+1}Z; R)$ q>0,r>0

such that for a section $Z \subset \Lambda_0 JZ$ and $\rho \in \Lambda_q(J^r Z; R)$:

$(j^r s)^* \rho = (j^{r+1}s)^* h\rho$

3) Thus the variational derivative computes as :

$\frac{d}{d\tau} S(\tau, Y)|_{\tau=0} = \int_\Omega (j^1 Z)^* \pounds_{j^1 Y} \mathcal{L}\varpi_0 = \int_\Omega (j^1 Z)^* (Y^i E_i \varpi_0 + hd(i_{j^1 Y}\Theta_L))$

But $\int_\Omega (j^1 Z)^* hd(i_{j^1 Y}\Theta_L) = \int_U Z^* d i_{j^1 Y}\Theta_L = \int_{\partial U} Z^* i_{j^1 Y}\Theta_L = 0$ with the Stockes theorem if Y is compactly supported

So for the solutions:

$\forall Y^i : \frac{d}{d\tau} S(\tau, Y)|_{\tau=0} = \int_\Omega (j^1 Z)^* Y^i E_i \varpi_0 = 0 \Rightarrow E_i = \frac{\partial \mathcal{L}}{\partial z^i} - \sum_{\alpha=1}^n \frac{d}{d\xi^\alpha}\left(\frac{\partial \mathcal{L}}{\partial z^i_\alpha}\right) = 0$

and we have the Euler-Lagrange equations:

$\frac{\partial \mathcal{L}}{\partial z^i} - \sum_{\alpha=1}^n \frac{d}{d\xi^\alpha}\left(\frac{\partial \mathcal{L}}{\partial z^i_\alpha}\right) = \mathbf{0}$

4) This classical method can be implemented for the "field part" $\mathcal{L}_F$ of our lagrangian, but in the "matter part" $\mathcal{L}_M$ the map $\widetilde{f}$ does not fit well. So we tackle the problem through the method of functional derivatives.

7.2 Functional derivatives

1) Let A be a set of scalar valued functions endowed with a Banach vector space structure. A functional is a continuous operator : $S : A \to \mathbb{C}$. The general theory of derivatives can be fully implemented. The functional derivative of S at f is a linear map :$\frac{dS}{df}(f) : A \to \mathbb{C}$ such that for any infinitesimal $\delta f : S(f + \delta f) - S(f) = \frac{dS}{df}(f)(\delta f) + o(\delta f)\|\delta f\|$

It is computed easily by $\frac{\delta S}{\delta f}(\delta f) = \frac{d}{d\tau} S(f + \tau \delta f)|_{\tau=0}$ where δf is a compactly supported function. $\frac{\delta S}{\delta f}$ is a distribution if it is continuous. We have the usual theorems and properties of derivatives with some caution because the product of two distributions is not defined. What matters is not the domain where the function are defined, but the codomain, where they take their value : the theory is legitimate as long as A is a Banach vector space, such that functions can be added together.

2) If both F and f belong to A, the chain rule gives : $\frac{dS}{df}(F\circ f)=\frac{dS}{dF}(F\circ f)\frac{dF}{df}$. As the product of the function $\frac{dF}{df}$ by the distribution $\frac{\delta S}{\delta F}(F)$ is well defined : $\frac{\delta S}{\delta f}(F\circ f)(\delta f)=\frac{dF}{df}\frac{\delta S}{\delta F}(F\circ f)(\delta f)$. One can compute the derivatives simultaneously with respect to F and f. Let us consider the function : $\phi(\tau_1,\tau_2)=S((F+\tau_1\delta F)\circ(f+\tau_2\delta f))$ and its partial derivatives with respect to τ_1,τ_2 in $\tau_1=0,\tau_2=0$. That is :

$\delta S=S((F+\tau_1\delta F)\circ(f+\tau_2\delta f))-S(F\circ f)=\phi(\tau_1,\tau_2)-\phi(0,0)$

$=\left(\frac{\partial\phi}{\partial\tau_1}|_{\tau_1,\tau_2=0}\right)\tau_1+\left(\frac{\partial\phi}{\partial\tau_2}|_{\tau_1,\tau_2=0}\right)\tau_2+o(\tau_1,\tau_2)(|\tau_1|+|\tau_2|)$

$\frac{\partial\phi}{\partial\tau_1}|_{\tau_1=0}=\frac{\delta S}{\delta F}(F\circ(f+\tau_2\delta f))(\delta F\circ(f+\tau_2\delta f))$

$\Rightarrow\frac{\partial\phi}{\partial\tau_1}|_{\tau_1,\tau_2=0}=\frac{\delta S}{\delta F}(F\circ f)\delta(F\circ f)$

$\frac{\partial\phi}{\partial\tau_2}|_{\tau_2=0}=\frac{dF}{df}\frac{\delta S}{\delta F}((F+\tau_1\delta F)\circ f)\delta f$

$\Rightarrow\frac{\partial\phi}{\partial\tau_2}|_{\tau_1,\tau_2=0}=\frac{dF}{df}\frac{\delta S}{\delta F}(F\circ f)\delta f$

Thus : $\delta S=\frac{\delta S}{\delta F}(F\circ f)\tau_1\delta(F\circ f)+\frac{dF}{df}\frac{\delta S}{\delta F}(F\circ f)\tau_2(\delta f)+o(\tau_1,\tau_2)(|\tau_1|+|\tau_2|)$

The functional derivative of S with respect to $F\circ f$ is : $\frac{\delta S}{\delta F}=\frac{\delta S}{\delta F}(F\circ f)$;

and the functional derivative of S with respect to f is : $\frac{\delta S}{\delta f}=\frac{dF}{df}\frac{\delta S}{\delta F}(F\circ f)$

Now if $S(F\circ f,f)$ we have by the same calculation :

$\delta S=S((F+\tau_1\delta F)\circ(f+\tau_2\delta f),f+\tau_2\delta f)-S(F\circ f)$

$\frac{\partial\phi}{\partial\tau_1}|_{\tau_1=0}=\frac{\delta S}{\delta F}(F\circ(f+\tau_2\delta f))(\delta F\circ(f+\tau_2\delta f),f+\tau_2\delta f)$

$\Rightarrow\frac{\partial\phi}{\partial\tau_1}|_{\tau_1,\tau_2=0}=\frac{\delta S}{\delta F}((F\circ f),f)\delta(F\circ f)$

$\frac{\partial\phi}{\partial\tau_2}|_{\tau_2=0}=\frac{dF}{df}\frac{\delta S}{\delta F}((F+\tau_1\delta F)\circ f,f)\delta f+\frac{\delta S}{\delta f}((F+\tau_1\delta F)\circ f,f)\delta f$

$\Rightarrow\frac{\partial\phi}{\partial\tau_2}|_{\tau_1,\tau_2=0}=\frac{dF}{df}\frac{\delta S}{\delta F}(F\circ f,f)\delta f+\frac{\delta S}{\delta f}(F\circ f,f)\delta f$

The functional derivative of S with respect to $F\circ f$ is : $\frac{\delta S}{\delta F}=\frac{\delta S}{\delta F}(F\circ f)$; and the functional derivative of S with respect to f is : $\frac{dF}{df}\frac{\delta S}{\delta F}(F\circ f,f)+\frac{\delta S}{\delta f}(F\circ f,f)$ where the last term is the functional derivative for f as a stand alone function.

3) Let us come back to our problem. JZ is a vector bundle, a section is valued in $\mathbb{R}^{16m+36}$, J^1Z is also a vector space and we will assume that we restrict ourselves to some set $H\subset\Lambda_0J^1Z$ of bounded, differentiable functions, endowed with a metric so that H is some Banach vector space. The functional derivative in j^1Z of a functional : $S:H\to\mathbb{R}$ is a continuous linear map : $U\in L(H;\mathbb{R})$ such that for any infinitesimal variation $j^1\delta Z=\left\{(\delta z^i,\delta z^i_\alpha)_{i,\alpha}\right\}\in J^1Z$:

$S(z^i+\delta z^i,z^i_\alpha+\delta z^i_\alpha)-S(z^i,z^i_\alpha)=U(\delta z^i,\delta z^i_\alpha)+o(\delta z^i,\delta z^i_\alpha)\|(\delta z^i,\delta z^i_\alpha)\|$

For a section $j^1Z \in H$ and the variation $\delta j^1 Z = (\delta z^i, \partial_\alpha \delta z^i) = j^1 \delta Z$ it reads :

$S(z^i+\delta z^i, \partial_\alpha z^i+\partial_\alpha \delta z^i)-S(z^i, \partial_\alpha z^i) = U(\delta z^i, \partial_\alpha \delta z^i)+o(\delta z^i, \partial_\alpha \delta z^i)\|(\delta z^i, \partial_\alpha \delta z^i)\|$

that is : $S(j^1\delta Z) - S(j^1 Z) = U(j^1\delta Z) + o(j^1\delta Z)\|(j^1\delta Z)\|$

If $\delta j^1 Z$ is defined by a projectable vector field in each point m, we have for $\tau \in \mathbb{R}$:

$j^1_m \delta Z = \Phi_\tau^{j^1 Y}(j^1_m Z) - j^1_m Z = \tau \left(\frac{d}{d\tau}\Phi_\tau^{j^1Y}(j^1_m Z)|_{\tau=0}\right) + \tau o(\tau) = \tau j^1 Y(j^1_m Z) + \tau o(\tau)$

Therefore the functional derivative must meet the condition :

$S(\Phi_\tau^{j^1Y}(j^1_m Z)) - S(j^1_m Z) = \tau U j^1 Y(j^1_m Z) + \tau o(\tau)$

The variational derivative is $\frac{d}{d\tau}S(\tau, Y)|_{\tau=0} = \int_\Omega (j^1Z)^* Y^i E_i \varpi_0$ thus it can be associated with the functional derivative :

$\frac{\delta S}{\delta z^i}(j^1Z) = (j^1Z)^* E_i$ such that :

$S(\Phi_\tau^{j^1Y}(j^1_m Z)) - S(j^1_m Z) = \tau \int_\Omega (j^1Z)^* Y^i E_i \varpi_0 + \tau o(\tau)$

The variational derivative is nothing but the value of the functional derivative along a projectable vector field. Furthermore :

$\sum_i \frac{\delta S}{\delta z^i}(j^1Z) Y^i \varpi_0 = (j^1Z)^* \pounds_{j^1Y} \mathcal{L} \varpi_0$

4) To implement this method we need to come back to functions with codomain $\mathbb{R}^n$. The trivialization of $\Omega : \varphi_\Omega : S(0) \times [0,T] \to \Omega$ stems from a chart : $\varphi_U : U_0 \times [0,T] \to \Omega$ where U_0 is an open set in $\mathbb{R}^3$. Let us define the map :

$f : U_0 \times [0,T] \to U_0 \times [0,T] :: f = \varphi_U^{-1} \circ \widetilde{f} \circ \varphi_U$

$\alpha = 0,1,23 : \eta^\alpha = f^\alpha(\xi^1, \xi^2, \xi^3, t)$ are the coordinates at t of a particle whith coordinates $(\xi^1, \xi^2, \xi^3, 0)$ at t=0 and $V^\alpha = \frac{\partial f^\alpha}{\partial t}$

The matter part of the action is the functionnal (in putting V apart)

$S_M = \int_\Omega N(m) L_M\left(V, z^i \circ \widetilde{f}, z^i_\alpha \circ \widetilde{f}\right)\varpi_4 =$

$\int_{U_0\times[0,T]} (N \circ \varphi_U) L_M\left(\frac{\partial f^\alpha}{\partial t}, z^i \widetilde{f}\varphi_U, z^i_\alpha \widetilde{f}\varphi_U\right)\left|\det O'\left(\widetilde{f}\varphi_U\right)\right|(\xi)\, d\xi^1 \otimes d\xi^2 \otimes d\xi^3 \otimes dt$

$S_M = \int_{U_0\times[0,T]} (N \circ \varphi_U) L\left(\frac{\partial f^\alpha}{\partial t}, z^i \varphi_U f, z^i_\alpha \varphi_U f\right)|\det O'(\varphi_U f)|(\xi)\, d\xi^1 \otimes d\xi^2 \otimes d\xi^3 \otimes dt$

N is fixed and we have a functional of the composite function $j^1 Z^\diamond = j^1 Z \circ f$ of the vector valued functions : $f(\xi)$ and $j^1 Z = \left(z^i(\xi), \frac{dz^i}{d\xi^\alpha}(\xi)\right)$.

5) For the field part of the action, which does not depend on f, the functional derivative is :
$\frac{\delta S_F}{\delta z^i}\left(j^1Z\right) = \frac{\partial \mathcal{L}_F}{\partial z^i}\left(j^1Z\right) - \sum_\alpha \frac{d}{d\xi^\alpha}\left(\frac{\partial \mathcal{L}_F}{\partial z^i_\alpha}\left(j^1Z\right)\right) = \frac{\partial \mathcal{L}_F}{\partial z^i} - \sum_\alpha \partial_\alpha \left(\frac{\partial \mathcal{L}_F}{\partial z^i_\alpha}\left(j^1Z\right)\right)$

6) For the matter part the functional derivatives are :
a) For i>0 : $\frac{\delta S_M}{\delta z^i}\left(j^1Z^\diamond\right) = \frac{\delta S_M}{\delta z^i}\left(j^1Z^\diamond\right) = E_i\left(j^1Z^\Diamond\right) = \frac{\partial \mathcal{L}_M}{\partial z^i}\left(j^1Z^\Diamond\right) - \sum_\alpha \frac{d}{d\xi^\alpha}\left(\frac{\partial \mathcal{L}_M}{\partial z^i_\alpha}\left(j^1Z^\Diamond\right)\right)$

The quantities $\frac{d}{d\xi^\alpha}\left(\frac{\partial \mathcal{L}_M}{\partial z^i_\alpha}\left(j^1Z^\Diamond\right)\right)$ must be read with the total derivatives :
$\frac{d}{d\xi^\alpha}\left(\frac{\partial \mathcal{L}_M}{\partial z^i_\alpha}\left(j^1Z^\Diamond\right)\right) = \sum_j \frac{\partial \mathcal{L}_M}{\partial z^j \partial z^i_\alpha}\frac{dz^j}{d\xi^\alpha} + \sum_{j\beta} \frac{\partial \mathcal{L}_M}{\partial z^j_\beta \partial z^i_\alpha}\frac{dz^j_\beta}{d\xi^\alpha}$
b) for f :
as a stand alone function :
$\frac{\delta S_M}{\delta f^\alpha} = -\sum_\beta \frac{d}{d\xi^\beta}\left(\frac{\partial \mathcal{L}_M}{\partial f^\alpha_\beta}\right) = -\frac{d}{d\xi^0}\left(\frac{\partial \mathcal{L}_M}{\partial V^\alpha}\right)$
and composed with the other functions :
$\frac{\delta S_M}{\delta f^\alpha}\left(j^1Z^\diamond\right) = \sum_{i>0}\frac{dz^i}{df^\alpha}\frac{\delta S_M}{\delta z^i}\left(j^1Z^\diamond\right) - \frac{d}{d\xi^0}\left(\frac{\partial \mathcal{L}_M}{\partial V^\alpha}\right)$ with : $\frac{dz^i}{df^\alpha} = \partial_\alpha z^i$

7) The functional derivative of $S = S_M + S_F$ is the sum of the functional derivatives :
i > 0

$$\frac{\delta S}{\delta z^i}=\frac{\partial \mathcal{L}_M}{\partial z^i}\left(j^1Z^\Diamond\right) - \sum_\alpha \frac{d}{d\xi^\alpha}\left(\frac{\partial \mathcal{L}_M}{\partial z^i_\alpha}\left(j^1Z^\Diamond\right)\right) + \frac{\partial \mathcal{L}_F}{\partial z^i}\left(j^1Z\right) - \sum_\alpha \partial_\alpha\left(\frac{\partial \mathcal{L}_F}{\partial z^i_\alpha}\left(j^1Z\right)\right) \tag{48}$$

and for f :

$$\frac{\delta S}{\delta f^\alpha}\left(j^1Z^\diamond\right) = \sum_{i>0}\left(\partial_\alpha z^i\right)\frac{\delta S_M}{\delta z^i}\left(j^1Z^\diamond\right) - \frac{d}{d\xi^0}\left(\frac{\partial \mathcal{L}_M}{\partial V^\alpha}\right) \tag{49}$$

We have furthermore for any projectable vector field Y:
$\sum_i \frac{\delta S}{\delta z^i}\left(j^1 s\right) Y^i \varpi_0 = \left(j^1 s\right)^* \pounds_{j^1Y}\left(\mathcal{L}\varpi_0\right) = \left(j^1 s\right)^* \pounds_{j^1Y}\left(\mathcal{L}^\Diamond_M + \mathcal{L}_F\right)\varpi_0$

8 LAGRANGE EQUATIONS

8.1 Equation of state

The equations for the state ψ are :

$\forall i,j: \frac{\delta S}{\delta \operatorname{Re}\psi^{ij}} = N(m)\frac{dL_M(\det O')}{d\operatorname{Re}\psi^{ij}}\left(Z^\Diamond\right) - \sum_\beta \frac{d}{d\xi^\beta}\left(N(m)\frac{dL_M(\det O')}{d\partial_\beta \operatorname{Re}\psi^{ij}}\left(Z^\Diamond\right)\right)$

$\forall i,j: \frac{\delta S}{\delta \operatorname{Im}\psi^{ij}} = N(m)\frac{dL_M(\det O')}{d\operatorname{Im}\psi^{ij}}\left(Z^\Diamond\right) - \sum_\beta \frac{d}{d\xi^\beta}\left(N(m)\frac{dL_M(\det O')}{d\partial_\beta \operatorname{Im}\psi^{ij}}\left(Z^\Diamond\right)\right)$

With :

$\frac{d\mathcal{L}_M}{d\operatorname{Re}\psi^{ij}} = \frac{\partial\mathcal{L}_M}{\partial\operatorname{Re}\psi^{ij}} + \sum_k \frac{d\mathcal{L}_M}{d\operatorname{Re}\nabla_\alpha\psi^{kj}}\operatorname{Re}\left[G_\alpha\right]_i^k + \frac{d\mathcal{L}_M}{d\operatorname{Im}\nabla_\alpha\psi^{kj}}\operatorname{Im}\left[G_\alpha\right]_i^k$
$+\frac{d\mathcal{L}_M}{d\operatorname{Re}\nabla_\alpha\psi^{ik}}\operatorname{Re}\left[\grave{A}_\alpha\right]_j^k + \frac{d\mathcal{L}_M}{d\operatorname{Im}\nabla_\alpha\psi^{ik}}\operatorname{Im}\left[\grave{A}_\alpha\right]_j^k$

$\frac{d\mathcal{L}_M}{d\partial_\beta\operatorname{Re}\psi^{ij}} = \frac{d\mathcal{L}_M}{d\nabla_\beta\operatorname{Re}\psi^{ij}}$

$\frac{d\mathcal{L}_M}{d\operatorname{Im}\psi^{ij}} = \frac{\partial\mathcal{L}_M}{\partial\operatorname{Im}\psi^{ij}} + \sum_k -\frac{d\mathcal{L}_M}{d\operatorname{Re}\nabla_\alpha\psi^{kj}}\operatorname{Im}\left[G_\alpha\right]_i^k + \frac{d\mathcal{L}_M}{d\operatorname{Im}\nabla_\alpha\psi^{kj}}\operatorname{Re}\left[G_\alpha\right]_i^k$
$-\frac{d\mathcal{L}_M}{d\operatorname{Re}\nabla_\alpha\psi^{ik}}\operatorname{Im}\left[\grave{A}_\alpha\right]_j^k + \frac{d\mathcal{L}_M}{d\operatorname{Im}\nabla_\alpha\psi^{ik}}\operatorname{Re}\left[\grave{A}_\alpha\right]_j^k$

$\frac{d\mathcal{L}_M}{d\partial_\beta\operatorname{Im}\psi^{ij}} = \frac{d\mathcal{L}_M}{d\nabla_\beta\operatorname{Im}\psi^{ij}}$

we get the equations :

$$\forall i,j: 0 = V\frac{\partial\mathcal{L}_M^\Diamond}{\partial\operatorname{Re}\psi^{ij}} + \sum_{\alpha k}\frac{d\mathcal{L}_M^\Diamond}{d\operatorname{Re}\nabla_\alpha\psi^{kj}}\operatorname{Re}\left[G_\alpha\right]_i^k + \frac{d\mathcal{L}_M^\Diamond}{d\operatorname{Im}\nabla_\alpha\psi^{kj}}\operatorname{Im}\left[G_\alpha\right]_i^k$$
$$+\frac{d\mathcal{L}_M^\Diamond}{d\operatorname{Re}\nabla_\alpha\psi^{ik}}\operatorname{Re}\left[\grave{A}_\alpha\right]_j^k + \frac{d\mathcal{L}_M^\Diamond}{d\operatorname{Im}\nabla_\alpha\psi^{ik}}\operatorname{Im}\left[\grave{A}_\alpha\right]_j^k\} - \sum_\beta\frac{d}{d\xi^\beta}\left(\frac{d\mathcal{L}_M^\Diamond}{d\operatorname{Re}\nabla_\beta\psi^{ij}}\right) \tag{50}$$

$$\forall i,j: 0 = \frac{\partial\mathcal{L}_M^\Diamond}{\partial\operatorname{Im}\psi^{ij}} + \sum_{\alpha k} -\frac{d\mathcal{L}_M^\Diamond}{d\operatorname{Re}\nabla_\alpha\psi^{kj}}\operatorname{Im}\left[G_\alpha\right]_i^k + \frac{d\mathcal{L}_M^\Diamond}{d\operatorname{Im}\nabla_\alpha\psi^{kj}}\operatorname{Re}\left[G_\alpha\right]_i^k$$
$$-\frac{d\mathcal{L}_M^\Diamond}{d\operatorname{Re}\nabla_\alpha\psi^{ik}}\operatorname{Im}\left[\grave{A}_\alpha\right]_j^k + \frac{d\mathcal{L}_M^\Diamond}{d\operatorname{Im}\nabla_\alpha\psi^{ik}}\operatorname{Re}\left[\grave{A}_\alpha\right]_j^k - \sum_\beta\frac{d}{d\xi^\beta}\left(\frac{d\mathcal{L}_M^\Diamond}{d\operatorname{Im}\nabla_\alpha\psi^{ij}}\right) \tag{51}$$

8.2 Gravitational equations

The equations for the gravitational potential G are :

$\forall a,\alpha: \frac{\delta S}{\delta G_\alpha^a} = \frac{d\mathcal{L}_M}{dG_\alpha^a}\left(Z^\Diamond\right) + \frac{d\mathcal{L}_F}{dG_\alpha^a}\left(Z(m)\right) - \sum_\beta\frac{d}{d\xi^\beta}\left(\frac{d\mathcal{L}_M}{d\partial_\beta G_\alpha^a}\left(Z^\Diamond\right) + \frac{d\mathcal{L}_F}{d\partial_\beta G_\alpha^a}\left(Z(m)\right)\right)$

$\frac{d\mathcal{L}_M}{dG^a_\alpha} = \frac{\partial \mathcal{L}_M}{\partial G^a_\alpha} + \sum_{ij} \left(\frac{d\mathcal{L}_M}{d \operatorname{Re} \partial_\alpha \psi^{ij}} \operatorname{Re} \left([\kappa_a] \left[\psi^\Diamond \right] \right)^i_j + \frac{d\mathcal{L}_M}{d \operatorname{Im} \partial_\alpha \psi^{ij}} \operatorname{Im} \left([\kappa_a] \left[\psi^\Diamond \right] \right)^i_j \right)$
$\frac{d\mathcal{L}_F}{dG^a_\alpha} = \frac{\partial \mathcal{L}_F}{\partial G^a_\alpha} + 2 \sum_{b\beta} \frac{d\mathcal{L}_F}{d\mathcal{F}^b_{G\alpha\beta}} \left[\overrightarrow{\kappa}_a, G_\beta \right]^b$
$\frac{d\mathcal{L}_M}{d\partial_\beta G^a_\alpha} = 0$
$\frac{d\mathcal{L}_F}{d\partial_\beta G^a_\alpha} = \sum_{\mu\lambda} \frac{dL_F}{d\mathcal{F}^b_{G\lambda\mu}} \frac{d\mathcal{F}^b_{G\lambda\mu}}{d\partial_\beta G^a_\alpha} = \sum_{\mu\lambda} \frac{dL_F}{d\mathcal{F}^b_{G\lambda\mu}} \frac{d\left(\partial_\lambda G^b_\mu - \partial_\mu G^b_\lambda \right)}{d\partial_\beta G^a_\alpha}$
$= \frac{dL_F}{d\mathcal{F}^a_{G\beta\alpha}} - \frac{dL_F}{d\mathcal{F}^a_{G\alpha\beta}} = 2 \frac{d\mathcal{L}_F}{d\mathcal{F}^a_{G\beta\alpha}}$
$\frac{d}{d\xi^\beta} \frac{d\mathcal{L}_F}{d\partial_\beta G^a_\alpha} (Z(m)) = \partial_\beta \frac{d\mathcal{L}_F}{d\partial_\beta G^a_\alpha}$
which gives the equations :

$$\forall a, \alpha : 0 = \frac{\partial \mathcal{L}^\Diamond_M}{\partial G^a_\alpha} + \frac{\partial \mathcal{L}_F}{\partial G^a_\alpha} + \sum_{ij} \left(\frac{\partial \mathcal{L}^\Diamond_M}{\partial \operatorname{Re} \nabla_\alpha \psi^{ij}} \operatorname{Re} \left([\kappa_a] [\psi] \right)^i_j + \frac{\partial \mathcal{L}^\Diamond_M}{\partial \operatorname{Im} \nabla_\alpha \psi^{ij}} \operatorname{Im} \left([\kappa_a] [\psi] \right)^i_j \right)$$
$$+ 2 \sum_\beta \left(\sum_b \frac{\partial \mathcal{L}_F}{\partial \mathcal{F}^b_{G\alpha\beta}} \left[\overrightarrow{\kappa}_a, G_\beta \right]^b + \partial_\beta \left(\frac{\partial \mathcal{L}_F}{\partial \mathcal{F}^a_{G\alpha\beta}} \right) \right) \tag{52}$$

8.3 Equations for the other force fields

The equations for the other potentials À are :

1) $\forall a, \alpha : \frac{\delta S}{\delta \operatorname{Re} \grave{A}^a_\alpha} = \frac{d\mathcal{L}^\Diamond_M}{d \operatorname{Re} \grave{A}^a_\alpha} \left(Z^\Diamond \right) + \frac{d\mathcal{L}_F}{d \operatorname{Re} \grave{A}^a_\alpha} (Z(m)) - \sum_\beta \frac{d}{d\xi^\beta} \left(\frac{d\mathcal{L}_F}{d\partial_\beta \operatorname{Re} \grave{A}^a_\alpha} (Z(m)) \right) = 0$

with
$\frac{d\mathcal{L}_M}{d \operatorname{Re} \grave{A}^a_\alpha} = \sum_{ij} \frac{d\mathcal{L}_M}{d \operatorname{Re} \nabla_\alpha \psi^{ij}} \operatorname{Re} \left(\left[\psi^\Diamond \right] [\theta_a]^t \right)^i_j + \frac{d\mathcal{L}_M}{d \operatorname{Im} \nabla_\alpha \psi^{ij}} \operatorname{Im} \left(\left[\psi^\Diamond \right] [\theta_a]^t \right)^i_j$
$\frac{d\mathcal{L}_F}{d \operatorname{Re} \grave{A}^a_\alpha} = 2 \sum_{b\beta} \frac{d\mathcal{L}_F}{d \operatorname{Re} \mathcal{F}^b_{G\alpha\beta}} \operatorname{Re} \left[\overrightarrow{\theta}_a, \grave{A}_\beta \right]^b + \frac{d\mathcal{L}_F}{d \operatorname{Im} \mathcal{F}^b_{G\alpha\beta}} \operatorname{Im} \left[\overrightarrow{\theta}_a, \grave{A}_\beta \right]^b$
$\frac{d\mathcal{L}_F}{d\partial_\beta \operatorname{Re} \grave{A}^a_\alpha} = 2 \frac{d\mathcal{L}_F}{d \operatorname{Re} \mathcal{F}^a_{\beta\alpha}}$
we get the equation :

$$\forall a, \alpha : 0 = \sum_{ij} \left(\frac{\partial \mathcal{L}^\Diamond_M}{\partial \operatorname{Re} \nabla_\alpha \psi^{ij}} \operatorname{Re} \left([\psi] [\theta_a]^t \right)^i_j + \frac{\partial \mathcal{L}^\Diamond_M}{\partial \operatorname{Im} \nabla_\alpha \psi^{ij}} \operatorname{Im} \left([\psi] [\theta_a]^t \right)^i_j \right)$$
$$+ 2 \sum_\beta \sum_b \left(\frac{\partial \mathcal{L}_F}{\partial \operatorname{Re} \mathcal{F}^b_{A,\alpha\beta}} \operatorname{Re} \left[\overrightarrow{\theta}_a, \grave{A}_\beta \right]^b + \frac{\partial \mathcal{L}_F}{\partial \operatorname{Im} \mathcal{F}^b_{A,\alpha\beta}} \operatorname{Im} \left[\overrightarrow{\theta}_a, \grave{A}_\beta \right]^b \right) + \partial_\beta \left(\frac{\partial \mathcal{L}_F}{\partial \operatorname{Re} \mathcal{F}^a_{A,\alpha\beta}} \right) \tag{53}$$

2) The second set of equations is :

$\forall a,\alpha:\frac{\delta S}{\delta\operatorname{Im}\grave{A}_{\alpha}^{a}}=\frac{\partial\mathcal{L}_{M}^{\Diamond}}{\partial\operatorname{Im}\grave{A}_{\alpha}^{a}}\left(Z^{\Diamond}\right)+\frac{\partial L_{F}(\det O')}{\partial\operatorname{Im}\grave{A}_{\alpha}^{a}}\left(Z\left(m\right)\right)-\sum_{\beta}\partial_{\beta}\left(\frac{\partial L_{F}(\det O')}{\partial\partial_{\beta}\operatorname{Im}\grave{A}_{\alpha}^{a}}\left(Z\left(m\right)\right)\right)=0$

$\frac{d\mathcal{L}_{F}}{d\operatorname{Im}\grave{A}_{\alpha}^{a}}=\frac{\partial\mathcal{L}_{F}}{\partial\operatorname{Im}\grave{A}_{\alpha}^{a}}+2\sum_{b\beta}-\frac{d\mathcal{L}_{F}}{d\operatorname{Re}\mathcal{F}_{G\alpha\beta}^{b}}\operatorname{Im}\left[\overrightarrow{\theta}_{a},\grave{A}_{\beta}\right]^{b}+\frac{d\mathcal{L}_{F}}{d\operatorname{Im}\mathcal{F}_{G\alpha\beta}^{b}}\operatorname{Re}\left[\overrightarrow{\theta}_{a},\grave{A}_{\beta}\right]^{b}$

$\frac{d\mathcal{L}_{M}}{d\operatorname{Im}\grave{A}_{\alpha}^{a}}=\sum_{ij}\left(-\frac{d\mathcal{L}_{M}}{d\operatorname{Re}\nabla_{\alpha}\psi^{ij}}\operatorname{Im}\left(\left[\psi^{\Diamond}\right]\left[\theta_{a}\right]^{t}\right)_{j}^{i}+\frac{d\mathcal{L}_{M}}{d\operatorname{Im}\nabla_{\alpha}\psi^{ij}}\operatorname{Re}\left(\left[\psi^{\Diamond}\right]\left[\theta_{a}\right]^{t}\right)_{j}^{i}\right)$

$\frac{d\mathcal{L}_{F}}{d\partial_{\beta}\operatorname{Im}\grave{A}_{\alpha}^{a}}=2\frac{d\mathcal{L}_{F}}{d\operatorname{Im}\mathcal{F}_{\beta\alpha}^{a}}$

and gives :

$$\begin{gathered}\forall a,\alpha:0=\sum_{ij}\left(-\frac{\partial\mathcal{L}_{M}^{\Diamond}}{\partial\operatorname{Re}\nabla_{\alpha}\psi^{ij}}\operatorname{Im}\left(\left[\psi\right]\left[\theta_{a}\right]^{t}\right)_{j}^{i}+\frac{\partial\mathcal{L}_{M}^{\Diamond}}{\partial\operatorname{Im}\nabla_{\alpha}\psi^{ij}}\operatorname{Re}\left(\left[\psi\right]\left[\theta_{a}\right]^{t}\right)_{j}^{i}\right)\\+2\sum_{\beta}\sum_{b}\left(-\frac{\partial\mathcal{L}_{F}}{\partial\operatorname{Re}\mathcal{F}_{A,\alpha\beta}^{b}}\operatorname{Im}\left[\overrightarrow{\theta}_{a},\grave{A}_{\beta}\right]^{b}+\frac{\partial\mathcal{L}_{F}}{\partial\operatorname{Im}\mathcal{F}_{A,\alpha\beta}^{b}}\operatorname{Re}\left[\overrightarrow{\theta}_{a},\grave{A}_{\beta}\right]^{b}\right)+\partial_{\beta}\left(\frac{\partial\mathcal{L}_{F}}{\partial\operatorname{Im}\mathcal{F}_{A,\alpha\beta}^{a}}\right)\end{gathered}\tag{54}$$

8.4 Frame equation

The equations for O' are :

$\forall i,\alpha:\frac{\delta S}{\delta O_{i}^{\prime\alpha}}=\frac{d\mathcal{L}}{dO_{\alpha}^{\prime i}}-\sum_{\gamma}\frac{d}{d\xi^{\gamma}}\left(\frac{d\mathcal{L}}{d\partial_{\gamma}O_{\alpha}^{\prime i}}\right)=0$

with $\mathcal{L}=NL_{M}^{\Diamond}\det O^{\prime\diamond}+L_{F}\det O'$

$\frac{\delta S}{\delta O_{i}^{\prime\alpha}}=N\left(m\right)\frac{dL_{M}(\det O')}{dO_{\alpha}^{\prime i}}\left(Z^{\Diamond}\right)+\frac{dL_{F}(\det O')}{dO_{\alpha}^{\prime i}}\left(Z\left(m\right)\right)$

$-\sum_{\gamma}\frac{d}{d\xi^{\gamma}}\left(N\frac{dL_{M}(\det O')}{d\partial_{\gamma}O_{\alpha}^{\prime i}}\left(Z^{\Diamond}\right)+\frac{dL_{F}(\det O')}{d\partial_{\gamma}O_{\alpha}^{\prime i}}\left(Z\left(m\right)\right)\right)$

with $\frac{\partial(\det O')}{\partial O_{\alpha}^{\prime i}}=O_{i}^{\alpha}\left(\det O'\right),L=VL_{M}^{\Diamond}+L_{F}$

$\frac{dL}{dO_{\alpha}^{\prime i}}+O_{i}^{\alpha}L-\frac{1}{\det O'}\sum_{\gamma}\left(\frac{d}{d\xi^{\beta}}\left(V\frac{dL\det O'}{d\partial_{\gamma}O_{\alpha}^{\prime i}}\right)\right)=0$

multiplying by $O_{\beta}^{\prime i}$ and adding :

$$\begin{gathered}0=\sum_{i}\frac{dL}{dO_{\alpha}^{\prime i}}O_{\beta}^{\prime i}+\delta_{\beta}^{\alpha}L-\frac{1}{\det O'}\sum_{i\gamma}O_{\beta}^{\prime i}\left(\frac{d}{d\xi^{\gamma}}\left(N\frac{dL\det O'}{d\partial_{\gamma}O_{\alpha}^{\prime i}}\right)\right)\\0=\sum_{i}\frac{dL}{dO_{\alpha}^{\prime i}}O_{\beta}^{\prime i}+\delta_{\beta}^{\alpha}L-\frac{1}{\det O'}O_{\beta}^{\prime i}\sum_{\gamma}\left(\frac{d}{d\xi^{\gamma}}\left(N\frac{\partial L_{M}(\det O')}{\partial\partial_{\gamma}O_{\alpha}^{\prime i}}\left(Z^{\Diamond}\right)\right)+\partial_{\gamma}\left(\frac{\partial L_{F}(\det O')}{\partial\partial_{\gamma}O_{\alpha}^{\prime i}}\right)\right)\\0=\left(\frac{dL}{dO_{\alpha}^{\prime i}}-\sum_{\gamma}\left(\frac{d}{d\xi^{\gamma}}\frac{NdL_{M}^{\Diamond}}{d\partial_{\gamma}O_{\alpha}^{\prime i}}+\partial_{\gamma}\frac{dL_{F}}{d\partial_{\gamma}O_{\alpha}^{\prime i}}\right)\right)O_{\beta}^{\prime i}+\delta_{\beta}^{\alpha}L\\-\frac{1}{\det O'}O_{\beta}^{\prime i}\sum_{\gamma}\left(\frac{NdL_{M}^{\Diamond}}{d\partial_{\gamma}O_{\alpha}^{\prime i}}\frac{d\det O^{\prime\diamond}}{d\xi^{\gamma}}+\frac{dL_{F}}{d\partial_{\gamma}O_{\alpha}^{\prime i}}\partial_{\gamma}\det O'\right)\end{gathered}\tag{55}$$

$\forall\alpha,\beta: 0=\delta^\alpha_\beta L+\sum_i\left(\frac{dL}{dO'^i_\alpha}-\sum_\gamma\left(\frac{d}{d\xi^\gamma}\frac{NdL^\diamond_M}{d\partial_\gamma O'^i_\alpha}+\partial_\gamma\frac{dL_F}{d\partial_\gamma O'^i_\alpha}\right)\right)O'^i_\beta$
$-\frac{1}{\det O'}O'^i_\beta\sum_\gamma\left(\frac{NdL^\diamond_M}{d\partial_\gamma O'^i_\alpha}\frac{d\det O'^\diamond}{d\xi^\gamma}+\frac{dL_F}{d\partial_\gamma O'^i_\alpha}\partial_\gamma\det O'\right)$

If the partial derivatives $\partial_\gamma O'^i_\alpha$ do not appear in the lagrangian we have the simple equation:
$\forall\alpha,\beta: 0=\sum_i\frac{dL}{dO'^i_\alpha}O'^i_\beta+\delta^\alpha_\beta L$

8.5 Trajectory

1) The equation for f is :
$\forall\alpha: 0=\frac{\delta S}{\delta f^\alpha}\left(j^1Z^\diamond\right)=\sum_{i>0}\left(\partial_\alpha z^i\right)\frac{\delta S_M}{\delta z^i}\left(j^1Z^\diamond\right)-\frac{d}{d\xi^0}\left(\frac{\partial\mathcal{L}_M}{\partial V^\alpha}\right)$
$=\sum_{ij}\frac{\delta S^\diamond_M}{\delta\operatorname{Re}\psi^{ij}}\frac{\partial\operatorname{Re}\psi^{ij}}{\partial f^\alpha}+\frac{\delta S^\diamond_M}{\delta\operatorname{Im}\psi^{ij}}\frac{\partial\operatorname{Im}\psi^{ij}}{\partial f^\alpha}$
$+\sum_{a,\beta}\frac{\delta S^\diamond_M}{\delta\operatorname{Re}\grave{A}^a_\beta}\frac{\partial\operatorname{Re}\grave{A}^a_\beta}{\partial f^\alpha}+\frac{\delta S^\diamond_M}{\delta\operatorname{Im}\grave{A}^a_\beta}\frac{\partial\operatorname{Im}\grave{A}^a_\beta}{\partial f^\alpha}+\frac{\delta S^\diamond_M}{\delta G^a_\beta}\frac{\partial G^a_\beta}{\partial f^\alpha}$
$+\sum_{i,\beta}\frac{\delta S^\diamond_M}{\delta O'^i_\beta}\frac{\partial O'^i_\beta}{\partial f^\alpha}-\frac{d}{d\xi^0}\left(\frac{\partial\mathcal{L}_M}{\partial V^\alpha}\right)$
$\frac{\partial\operatorname{Re}\psi^{ij}}{\partial f^\alpha}=\partial_\alpha\operatorname{Re}\psi^{ij},\frac{\partial\operatorname{Im}\psi^{ij}}{\partial f^\alpha}=\partial_\alpha\operatorname{Im}\psi^{ij},\frac{\partial\operatorname{Re}\grave{A}^a_\beta}{\partial f^\alpha}=\partial_\alpha\operatorname{Re}\grave{A}^a_\beta,$
$\frac{\partial\operatorname{Im}\grave{A}^a_\beta}{\partial f^\alpha}=\partial_\alpha\operatorname{Im}\grave{A}^a_\beta,\frac{\partial G^a_\beta}{\partial f^\alpha}=\partial_\alpha G^a_\beta,\frac{\partial O'^i_\beta}{\partial f^\alpha}=\partial_\alpha O'^i_\beta$
all these partial derivatives being evaluated at $j^1Z^\diamond=\widetilde{f}^*j^1Z$
On shell we have :
$\frac{\delta S^\diamond_M}{\delta\operatorname{Re}\psi^{ij}}=0,\frac{\delta S^\diamond_M}{\delta\operatorname{Im}\psi^{ij}}=0,\frac{\delta S^\diamond_M}{\delta\operatorname{Re}\grave{A}^a_\beta}+\frac{\delta S_F}{\delta\operatorname{Re}\grave{A}^a_\beta}=0,$
$\frac{\delta S^\diamond_M}{\delta\operatorname{Im}\grave{A}^a_\beta}+\frac{\delta S_F}{\delta\operatorname{Im}\grave{A}^a_\beta}=0,\frac{\delta S^\diamond_M}{\delta G^a_\beta}+\frac{\delta S_F}{\delta G^a_\beta}=0,\frac{\delta S^\diamond_M}{\delta O'^i_\beta}+\frac{\delta S_F}{\delta O'^i_\beta}=0$
So we have two possible formulations for the equation. As L_F does not involve f, it is simpler to take its derivatives whenever useful.

2) $\frac{\delta S}{\delta f^\alpha}\left(j^1Z^\diamond\right)$
$=\left(\sum_{a,\beta}\frac{\delta S^\diamond_M}{\delta\operatorname{Re}\grave{A}^a_\beta}\partial_\alpha\operatorname{Re}\grave{A}^a_\beta+\frac{\delta S^\diamond_M}{\delta\operatorname{Im}\grave{A}^a_\beta}\partial_\alpha\operatorname{Im}\grave{A}^a_\beta+\frac{\delta S^\diamond_M}{\delta G^a_\beta}\partial_\alpha G^a_\beta+\sum_{i,\beta}\frac{\delta S^\diamond_M}{\delta O'^i_\beta}\partial_\alpha O'^i_\beta\right)\left(j^1Z^\diamond\right)-$
$\frac{d}{d\xi^0}\left(\frac{\partial\mathcal{L}_M}{\partial V^\alpha}\right)$
$=-\left(\sum_{a,\beta}\frac{\delta S_F}{\delta\operatorname{Re}\grave{A}^a_\beta}\partial_\alpha\operatorname{Re}\grave{A}^a_\beta+\frac{\delta S_F}{\delta\operatorname{Im}\grave{A}^a_\beta}\partial_\alpha\operatorname{Im}\grave{A}^a_\beta+\frac{\delta S_F}{\delta G^a_\beta}\partial_\alpha G^a_\beta+\sum_{i,\beta}\frac{\delta S_F}{\delta O'^i_\beta}\partial_\alpha O'^i_\beta\right)\left(j^1Z\right)-$
$\frac{d}{d\xi^0}\left(\frac{\partial\mathcal{L}_M}{\partial V^\alpha}\right)$
$\frac{\delta S_F}{\delta\operatorname{Re}\grave{A}^a_\beta}=2\sum_\gamma\{\sum_b\left(\frac{\partial\mathcal{L}_F}{\partial\operatorname{Re}\mathcal{F}^b_{A,\beta\gamma}}\operatorname{Re}\left[\overrightarrow{\theta}_a,\grave{A}_\gamma\right]^b+\frac{\partial\mathcal{L}_F}{\partial\operatorname{Im}\mathcal{F}^b_{A,\beta\gamma}}\operatorname{Im}\left[\overrightarrow{\theta}_a,\grave{A}_\gamma\right]^b\right)$

$+\partial_\gamma \left(\frac{\partial \mathcal{L}_F}{\partial \operatorname{Re} \mathcal{F}^a_{A,\beta\gamma}}\right)\}$

$\frac{\delta S_F}{\delta \operatorname{Im} \grave{A}^a_\beta} = 2\sum_\gamma \{\sum_b \left(-\frac{\partial \mathcal{L}_F}{\partial \operatorname{Re} \mathcal{F}^b_{A,\beta\gamma}} \operatorname{Im}\left[\overrightarrow{\theta}_a, \grave{A}_\gamma\right]^b + \frac{\partial \mathcal{L}_F}{\partial \operatorname{Im} \mathcal{F}^b_{A,\beta\gamma}} \operatorname{Re}\left[\overrightarrow{\theta}_a, \grave{A}_\gamma\right]^b\right)$

$+\partial_\gamma \left(\frac{\partial \mathcal{L}_F}{\partial \operatorname{Im} \mathcal{F}^b_{A,\beta\gamma}}\right)\}$

$\frac{\delta S_F}{\delta G^a_\beta} = \frac{\partial \mathcal{L}_F}{\partial G^a_\beta} + 2\sum_\gamma \left(\sum_b \frac{\partial \mathcal{L}_F}{\partial \mathcal{F}^b_{G\beta\gamma}} \left[\overrightarrow{\kappa}_a, G_\gamma\right]^b + \partial_\gamma \left(\frac{\partial \mathcal{L}_F}{\partial \mathcal{F}^a_{G\beta\gamma}}\right)\right)$

$\frac{\delta S_F}{\delta O'^i_\beta} = \frac{dL_F(\det O')}{dO'^i_\beta} - \sum_\gamma \frac{d}{d\xi^\gamma}\left(\frac{dL_F(\det O')}{d\partial_\gamma O'^i_\beta}\right)$

So on shell the equation is :

$\forall \alpha : 0 = \frac{d}{d\xi^0}\left(\frac{\partial \mathcal{L}_M}{\partial V^\alpha}\right)$

$+2\sum_{a,\beta\gamma}\{\sum_b \left(\frac{\partial \mathcal{L}_F}{\partial \operatorname{Re} \mathcal{F}^b_{A,\beta\gamma}} \operatorname{Re}\left[\overrightarrow{\theta}_a, \grave{A}_\gamma\right]^b + \frac{\partial \mathcal{L}_F}{\partial \operatorname{Im} \mathcal{F}^b_{A,\beta\gamma}} \operatorname{Im}\left[\overrightarrow{\theta}_a, \grave{A}_\gamma\right]^b\right)$

$+\partial_\gamma \left(\frac{\partial \mathcal{L}_F}{\partial \operatorname{Re} \mathcal{F}^a_{A,\beta\gamma}}\right) \partial_\alpha \operatorname{Re} \grave{A}^a_\beta\}$

$+2\sum_{a\beta\gamma}\{\sum_b \left(-\frac{\partial \mathcal{L}_F}{\partial \operatorname{Re} \mathcal{F}^b_{A,\beta\gamma}} \operatorname{Im}\left[\overrightarrow{\theta}_a, \grave{A}_\gamma\right]^b + \frac{\partial \mathcal{L}_F}{\partial \operatorname{Im} \mathcal{F}^b_{A,\beta\gamma}} \operatorname{Re}\left[\overrightarrow{\theta}_a, \grave{A}_\gamma\right]^b\right)$

$+\partial_\gamma \left(\frac{\partial \mathcal{L}_F}{\partial \operatorname{Im} \mathcal{F}^b_{A,\beta\gamma}}\right) \partial_\alpha \operatorname{Im} \grave{A}^a_\beta\}$

$+\sum_{a\beta} \left(\frac{\partial \mathcal{L}_F}{\partial G^a_\beta} + 2\sum_\gamma \left(\sum_b \frac{\partial \mathcal{L}_F}{\partial \mathcal{F}^b_{G\beta\gamma}} \left[\overrightarrow{\kappa}_a, G_\gamma\right]^b + \partial_\gamma \left(\frac{\partial \mathcal{L}_F}{\partial \mathcal{F}^a_{G\beta\gamma}}\right)\right)\right) \partial_\alpha G^a_\beta$

$+\sum_{i\beta} \left(\frac{dL_F(\det O')}{dO'^i_\beta} - \sum_\gamma \frac{d}{d\xi^\gamma}\left(\frac{dL_F(\det O')}{d\partial_\gamma O'^i_\beta}\right)\right) \partial_\alpha O'^i_\beta$

$\forall \alpha : 0 = \frac{d}{d\xi^0}\left(\frac{\partial \mathcal{L}_M}{\partial V^\alpha}\right) + 2\sum_{a,\beta\gamma} \left(\partial_\alpha \operatorname{Re} \grave{A}^a_\beta\right) \partial_\gamma \left(\frac{\partial \mathcal{L}_F}{\partial \operatorname{Re} \mathcal{F}^a_{A,\beta\gamma}}\right) + \left(\partial_\alpha \operatorname{Im} \grave{A}^a_\beta\right) \partial_\gamma \left(\frac{\partial \mathcal{L}_F}{\partial \operatorname{Im} \mathcal{F}^b_{A,\beta\gamma}}\right)$

$+2\sum_{b\beta\gamma} \frac{\partial \mathcal{L}_F}{\partial \operatorname{Re} \mathcal{F}^b_{A,\beta\gamma}} \operatorname{Re}\left(\left[\overrightarrow{\theta}_a, \grave{A}_\gamma\right]^b \partial_\alpha \grave{A}^a_\beta\right) + \frac{\partial \mathcal{L}_F}{\partial \operatorname{Im} \mathcal{F}^b_{A,\beta\gamma}} \operatorname{Im}\left(\left[\overrightarrow{\theta}_a, \grave{A}_\gamma\right]^b \partial_\alpha \grave{A}^a_\beta\right)\}$

$+\sum_{a\beta} \left(\frac{\partial \mathcal{L}_F}{\partial G^a_\alpha} \partial_\alpha G^a_\beta + 2\sum_\gamma \left(\left(\partial_\alpha G^a_\beta\right) \partial_\gamma \left(\frac{\partial \mathcal{L}_F}{\partial \mathcal{F}^a_{G\beta\gamma}}\right) + \frac{\partial \mathcal{L}_F}{\partial \mathcal{F}^a_{G\beta\gamma}} \left[\partial_\alpha G_\beta, G_\gamma\right]^a\right)\right)$

$+\sum_{i\beta} \left(\frac{dL_F(\det O')}{dO'^i_\beta} - \sum_\gamma \frac{d}{d\xi^\gamma}\left(\frac{dL_F(\det O')}{d\partial_\gamma O'^i_\beta}\right)\right) \partial_\alpha O'^i_\beta$

Thus the equation is :

$$(56)$$

$\forall \alpha : 0 = \frac{d}{d\xi^0}\left(\frac{\partial \mathcal{L}_M}{\partial V^\alpha}\right) + \sum_{i\beta} \left(\frac{dL_F(\det O')}{dO'^i_\beta} - \sum_\gamma \frac{d}{d\xi^\gamma}\left(\frac{dL_F(\det O')}{d\partial_\gamma O'^i_\beta}\right)\right) \partial_\alpha O'^i_\beta$

$+2\sum_{a,\beta\gamma}(\left(\partial_\alpha \operatorname{Re} \grave{A}^a_\beta\right) \partial_\gamma \left(\frac{\partial \mathcal{L}_F}{\partial \operatorname{Re} \mathcal{F}^a_{A,\beta\gamma}}\right) + \left(\partial_\alpha \operatorname{Im} \grave{A}^a_\beta\right) \partial_\gamma \left(\frac{\partial \mathcal{L}_F}{\partial \operatorname{Im} \mathcal{F}^b_{A,\beta\gamma}}\right)$

$+\frac{\partial \mathcal{L}_F}{\partial \operatorname{Re} \mathcal{F}^a_{A,\beta\gamma}} \operatorname{Re}\left[\partial_\alpha \grave{A}_\beta, \grave{A}_\gamma\right]^a + \frac{\partial \mathcal{L}_F}{\partial \operatorname{Im} \mathcal{F}^a_{A,\beta\gamma}} \operatorname{Im}\left[\partial_\alpha \grave{A}_\beta, \grave{A}_\gamma\right]^a)$

$+\sum_{a\beta} \left(\frac{\partial \mathcal{L}_F}{\partial G^a_\alpha} \partial_\alpha G^a_\beta + 2\sum_\gamma \left(\left(\partial_\alpha G^a_\beta\right) \partial_\gamma \left(\frac{\partial \mathcal{L}_F}{\partial \mathcal{F}^a_{G\beta\gamma}}\right) + \frac{\partial \mathcal{L}_F}{\partial \mathcal{F}^a_{G\beta\gamma}} \left[\partial_\alpha G_\beta, G_\gamma\right]^a\right)\right)$

9 NOETHER CURRENTS

9.1 Principles

For a one parameter group of diffeomorphisms with projectable vector field Y the Lie derivative is given by the formula 47 which is equivalent to the following (Krupka [15] p.44):

$\pounds_{j^1Y}\mathcal{L}\varpi_0 = \sum_i \left(\left(Y^i - z^i_\beta Y^\beta \right) E_i + \frac{d}{d\xi^\alpha} \left(Y^\alpha \mathcal{L} + \frac{\partial \mathcal{L}}{\partial z^i_\alpha} \left(Y^i - X^\beta z^i_\beta \right) \right) \right) \varpi_0$

that is if Y is vertical $(Y^\alpha = 0)$:

$\pounds_{j^1Y}\mathcal{L}\varpi_0 = \sum_i \left(Y^i E_i + \frac{d}{d\xi^\alpha} \left(\frac{\partial \mathcal{L}}{\partial z^i_\alpha} Y^i \right) \right) \varpi_0$

were $E_i(\mathcal{L}) = \frac{\partial \mathcal{L}}{\partial z^i} - \sum_\alpha \frac{d}{d\xi^\alpha} \left(\frac{\partial \mathcal{L}}{\partial z^i_\alpha} \right)$ are the Lagrange forms

and $\mathcal{L} = (VL_M + L_F) \det O'$.

On shell $E_i(\mathcal{L}) = 0$ so :$\sum_{i,\alpha} \frac{d}{d\xi^\alpha} \left(\frac{\partial \mathcal{L}}{\partial z^i_\alpha} Y^i \right) \varpi_0 = 0$

The equivariance implies for any vertical vector field parametrized by a section ζ of J^2F_M :

$\sum_i \frac{\partial \mathcal{L}_M}{\partial z^i} Y^i(Z,\zeta) + \frac{\partial \mathcal{L}_M}{\partial z^i_\alpha} Y^i_\alpha(Z,\zeta) = 0$

$\sum_i \frac{\partial \mathcal{L}_F}{\partial z^i} Y^i(Z,\zeta) + \frac{\partial \mathcal{L}_F}{\partial z^i_\alpha} Y^i_\alpha(Z,\zeta) = 0$

with $Y^i_\alpha(Z,\zeta) = \frac{d}{d\xi^\alpha} Y^i(Z,\zeta)$ that is : $\sum_i \frac{\partial \mathcal{L}}{\partial z^i} Y^i = -\sum_i \frac{\partial \mathcal{L}}{\partial z^i_\alpha} \frac{d}{d\xi^\alpha} Y^i$

Thus the quantity $\sum_i E_i(\mathcal{L}) Y^i = \sum_i \frac{\partial \mathcal{L}}{\partial z^i} Y^i - \sum_{i,\alpha} Y^i \frac{d}{d\xi^\alpha} \left(\frac{\partial \mathcal{L}}{\partial z^i_\alpha} \right)$ can be written :

$\sum_i E_i(\mathcal{L}) Y^i = -\sum_{i\alpha} \left(\frac{\partial \mathcal{L}}{\partial z^i_\alpha} Y^i_\alpha + Y^i \frac{d}{d\xi^\alpha} \left(\frac{\partial \mathcal{L}}{\partial z^i_\alpha} \right) \right)$

$= -\sum_{i\alpha} \left(\frac{\partial \mathcal{L}}{\partial \partial_\alpha z^i} \frac{dY^i}{d\xi^\alpha} + Y^i \frac{d}{d\xi^\alpha} \left(\frac{\partial \mathcal{L}}{\partial z^i_\alpha} \right) \right) = -\sum_\alpha \frac{d}{d\xi^\alpha} \left(\sum_i \frac{\partial \mathcal{L}}{\partial \partial_\alpha z^i} Y^i \right)$

and on shell we get for any vertical field Y : $\sum_\alpha \frac{d}{d\xi^\alpha} \left(\sum_i \frac{\partial \mathcal{L}}{\partial z^i_\alpha} Y^i \right) = 0$.. If one puts $\zeta = \left(-\overrightarrow{\kappa}_a, -\overrightarrow{\theta}_b \right)$ =Constant then for each generator of the gauge group $\sum_i \frac{\partial \mathcal{L}}{\partial \partial_\alpha z^i} Y^i \left(-\overrightarrow{\kappa}_a, -\overrightarrow{\theta}_b \right)$ is divergence free. But as we have seen only the partial derivatives such that $\frac{\partial L_M}{\partial z^i_\lambda}, \frac{\partial L_F}{\partial z^i_\lambda}$ are components of vector fields, so going to the conclusion is not so straightforward.

We will prove that both the gravitational equation and the equation for the other fields can be written in purely geometrical manner of the kind : $\varpi_4(Y) = \frac{1}{2} d\Pi$ where Y, the "Noether current", is a vector and Π , the "superpotential", is a 2-form.

9.2 Noether currents for the gravitational field

1) Let us fix $\overrightarrow{\kappa}=\sum_a \kappa^a \overrightarrow{\kappa}_a = Ct$ then the vector Y has the components :

$Y^{\operatorname{Re}\psi^{ij}}=\sum_a \kappa^a \operatorname{Re}\left([\kappa_a][\psi]\right)^i_j$

$Y^{\operatorname{Im}\psi^{ij}}=\sum_a \kappa^a \operatorname{Im}\left([\kappa_a][\psi]\right)^i_j$

$Y^{G^b_\beta}=\sum_a \kappa^a \left[\overrightarrow{\kappa}_a, G_\beta\right]^b$

$Y^{O'^i_\beta}=\sum_a \kappa^a \left([\widetilde{\kappa}_a][O']\right)^i_\beta$

V is not involved, so :

$\sum_i \frac{\partial \mathcal{L}}{\partial z^i_\alpha} Y^i(\overrightarrow{\kappa}) = \sum_{aij}\left(\frac{d\mathcal{L}}{d\operatorname{Re}\partial_\alpha \psi^{ij}}\kappa^a \operatorname{Re}\left([\kappa_a][\psi]\right)^i_j + \frac{d\mathcal{L}}{d\operatorname{Im}\partial_\alpha \psi^{ij}}\kappa^a \operatorname{Im}\left([\kappa_a][\psi]\right)^i_j\right)$

$+\sum_{ab\beta}\frac{d\mathcal{L}}{d\partial_\alpha G^b_\beta}\kappa^a\left[\overrightarrow{\kappa}_a, G_\beta\right]^b + \sum_{ai\beta}\frac{d\mathcal{L}}{d\partial_\alpha O'^i_\beta}\kappa^a\left([\widetilde{\kappa}_a][O']\right)^i_\beta$

$=\sum_a \kappa^a(\det O')\{\sum_{ij}\left(\frac{d(VL_M+L_F)}{d\operatorname{Re}\partial_\alpha\psi^{ij}}\operatorname{Re}\left([\kappa_a][\psi]\right)^i_j + \frac{d(VL_M+L_F)}{d\operatorname{Im}\partial_\alpha\psi^{ij}}\operatorname{Im}\left([\kappa_a][\psi]\right)^i_j\right)$

$+\sum_{b\beta}\frac{d(VL_M+L_F)}{d\partial_\alpha G^b_\beta}\left[\overrightarrow{\kappa}_a, G_\beta\right]^b + \sum_{i\beta}\frac{d(VL_M+L_F)}{d\partial_\alpha O'^i_\beta}\left([\widetilde{\kappa}_a][O']\right)^i_\beta\}$

with $\mathcal{L}=(NL_M+L_F)\det O'$

And :

$\frac{d(NL_M+L_F)}{d\operatorname{Re}\partial_\alpha\psi^{ij}} = N\frac{dL_M}{d\operatorname{Re}\nabla_\alpha\psi^{ij}}; \frac{dL}{d\operatorname{Im}\partial_\alpha\psi^{ij}} = N\frac{dL_M}{d\operatorname{Im}\nabla_\alpha\psi^{ij}}$

$\frac{d(VL_M+L_F)}{d\partial_\alpha G^a_\beta} = \frac{dL_F(\det O')}{d\partial_\alpha G^a_\alpha} = 2\frac{dL_F}{d\mathcal{F}^a_{G\alpha\beta}}$

$\frac{d(NL_M+L_F)}{d\partial_\alpha O'^i_\beta} = N\frac{dL_M}{d\partial_\alpha O'^i_\beta} + \frac{dL_F}{d\partial_\alpha O'^i_\beta}$

So :

$\sum_i \frac{\partial \mathcal{L}}{\partial z^i_\alpha} Y^i(\overrightarrow{\kappa})$

$=\sum_a \kappa^a(\det O')\{\sum_{ij} N\left(\frac{dL_M}{d\operatorname{Re}\nabla_\alpha\psi^{ij}}\operatorname{Re}\left([\kappa_a][\psi]\right)^i_j + \frac{dL_M}{d\operatorname{Im}\nabla_\alpha\psi^{ij}}\operatorname{Im}\left([\kappa_a][\psi]\right)^i_j\right)$

$+2\sum_{a,\beta}\frac{dL_F}{d\mathcal{F}^b_{G\alpha\beta}}\left[\overrightarrow{\kappa}_a, G_\beta\right]^b + \sum_{i\beta}\left(N\frac{dL_M}{d\partial_\alpha O'^i_\beta} + \frac{dL_F}{d\partial_\alpha O'^i_\beta}\right)\left([\widetilde{\kappa}_a][O']\right)^i_\beta\}$

The identities 27,38 from the gauge equivariance read :

$\forall a,\alpha : \frac{\partial L_F}{\partial G^a_\alpha} = \sum_{ij}[\widetilde{\kappa}_a]^i_j\sum_\beta \frac{dL_F}{d\partial_\alpha O'^i_\beta}O'^j_\beta$

$\forall a,\alpha : \frac{\partial L_M}{\partial G^a_\alpha} = \sum_{ij}[\widetilde{\kappa}_a]^i_j\sum_\beta \frac{dL_M}{d\partial_\alpha O'^i_{\beta\alpha}}O'^j_\beta$

$\sum_i \frac{\partial \mathcal{L}}{\partial z^i_\alpha} Y^i(\overrightarrow{\kappa})$

$=\sum_a \kappa^a(\det O')\{\sum_{ij} N\left(\frac{dL_M}{d\operatorname{Re}\nabla_\alpha\psi^{ij}}\operatorname{Re}\left([\kappa_a][\psi]\right)^i_j + \frac{dL_M}{d\operatorname{Im}\nabla_\alpha\psi^{ij}}\operatorname{Im}\left([\kappa_a][\psi]\right)^i_j\right)$

$+2\sum_{b,\beta}\frac{dL_F}{d\mathcal{F}^b_{G\alpha\beta}}\left[\overrightarrow{\kappa}_a, G_\beta\right]^b + V\frac{\partial L^\Diamond_M}{\partial G^a_\alpha} + \frac{\partial L_F}{\partial G^a_\alpha}\}$

But from the definition of the partial derivatives :

$N\sum_{ij}\left(\frac{dL^\Diamond_M}{d\operatorname{Re}\nabla_\alpha\psi^{ij}}\operatorname{Re}\left([\kappa_a][\psi]\right)^i_j + \frac{dL^\Diamond_M}{d\operatorname{Im}\nabla_\alpha\psi^{ij}}\operatorname{Im}\left([\kappa_a][\psi]\right)^i_j\right) + N\frac{\partial L^\Diamond_M}{\partial G^a_\alpha} = N\frac{dL^\Diamond_M}{dG^a_\alpha}$

$\frac{dL_F}{dG^a_\alpha} = \frac{\partial L_F}{\partial G^a_\alpha} + \sum_{\mu\lambda}\frac{dL_F}{d\mathcal{F}^b_{G\lambda\mu}}\frac{d\mathcal{F}^b_{G\lambda\mu}}{dG^a_\alpha}$

$= \frac{\partial L_F}{\partial G^a_\alpha} + \sum_{\mu\lambda} \frac{dL_F}{d\mathcal{F}^b_{G\lambda\mu}} \frac{d\left(G^b_{cd} G^c_\lambda G^d_\mu\right)}{dG^a_\alpha} = \frac{\partial L_F}{\partial G^a_\alpha} + 2\sum_{b\beta} \frac{dL_F}{d\mathcal{F}^b_{G\alpha\beta}} \left[\overrightarrow{\kappa}_a, G_\beta\right]^b$

So $\sum_i \frac{\partial \mathcal{L}}{\partial z^i_\alpha} Y^i\left(\overrightarrow{\kappa}\right) = \sum_a \kappa^a \left(\det O'\right) \left(N \frac{dL^\diamondsuit_M}{dG^a_\alpha} + \frac{dL_F}{dG^a_\alpha}\right) = \sum_a \kappa^a \frac{d\mathcal{L}}{dG^a_\alpha}$

The gravitational equation 52 reads:

$\forall a, \alpha : \frac{\delta S}{\delta G^a_\alpha} = \frac{d\mathcal{L}}{dG^a_\alpha} - \sum_\beta \frac{d}{d\xi^\beta} \left(\frac{d\mathcal{L}}{d\partial_\beta G^a_\alpha}\right) = 0$

so $\sum_i \frac{\partial \mathcal{L}}{\partial z^i_\alpha} Y^i\left(\overrightarrow{\kappa}\right) = \sum_a \kappa^a \left(\frac{\delta S}{\delta G^a_\alpha} + \sum_\beta \frac{d}{d\xi^\beta}\left(\frac{d\mathcal{L}}{d\partial_\beta G^a_\alpha}\right)\right)$ and on shell :

$\forall \overrightarrow{\kappa} : \sum_\alpha \frac{d}{d\xi^\alpha}\left(\sum_i \frac{\partial \mathcal{L}}{\partial z^i_\alpha} Y^i\left(\overrightarrow{\kappa}\right)\right) = \sum_a \kappa^a \sum_{\alpha\beta} \frac{d}{d\xi^\alpha}\frac{d}{d\xi^\beta}\left(\frac{d\mathcal{L}}{d\partial_\beta G^a_\alpha}\right) = 0$ because $\frac{d\mathcal{L}}{d\partial_\alpha G^a_\beta} + \frac{d\mathcal{L}}{d\partial_\beta G^a_\alpha} = 0$

2) $\frac{d(NL_M + L_F)}{d\partial_\beta G^a_\alpha} = \frac{dL_F}{d\partial_\beta G^a_\alpha} = 2\frac{dL_F}{d\mathcal{F}^a_{G,\beta\alpha}}$ are the components of the anti-symmetric bi-vector field : $Z^a_G = \sum_{\{\alpha\beta\}} \frac{dL_F}{d\partial_\alpha G^a_\beta} \partial_\alpha \wedge \partial_\beta$ and we can define the tensor : $Z_G = \sum_{a,\{\alpha\beta\}} \frac{dL_F}{d\partial_\alpha G^a_\beta} \partial_\alpha \wedge \partial_\beta \otimes \overrightarrow{\kappa}_a = 2\sum_{a,\{\alpha\beta\}} \frac{dL_F}{d\mathcal{F}^a_{G,\alpha\beta}} \partial_\alpha \wedge \partial_\beta \otimes \overrightarrow{\kappa}_a$. Beware that Z_G here and in the following is defined in respect with $\frac{dL_F}{d\partial_\alpha G^a_\beta}$ and the 2 factor is needed when Z_G is computed through $\frac{dL_F}{d\mathcal{F}^a_{G,\alpha\beta}}$.

We will compute the 2-form $\Pi_G = \varpi_4\left(Z_G\right)$ and its exterior differential. For this we first establish several formulas which will be extensively used in the following.

So let be the 2 antisymmetric bi-vector field : $Z = \sum_{\{\alpha\beta\}} Z^{\alpha\beta} \partial_\alpha \wedge \partial_\beta$

a) $\Pi = \varpi_4(Z) = \varpi_4\left(\sum_{\{\alpha\beta\}} Z^{\alpha\beta} \partial_\alpha \wedge \partial_\beta\right)$

$\varpi_0 = \sum_{\alpha_1\alpha_2\alpha_3\alpha_0} \epsilon\left(\alpha_0, \alpha_1, \alpha_2, \alpha_3\right) dx^{\alpha_0} \otimes dx^{\alpha_1} \otimes dx^{\alpha_2} \otimes dx^{\alpha_3}$

$Z = \sum_{\alpha<\beta a} Z^{\alpha\beta}\left(\partial_\alpha \otimes \partial_\beta - \partial_\beta \otimes \partial_\alpha\right)$

$\varpi_0(Z)$

$= \sum_{\alpha_1\alpha_2\alpha_3\alpha_0} \sum_{\alpha<\beta} Z^{\alpha\beta} \epsilon\left(\alpha_0, \alpha_1, \alpha_2, \alpha_3\right)$

$\{(dx^{\alpha_0} \otimes dx^{\alpha_1} \otimes dx^{\alpha_2}\left(\partial_\alpha\right) \otimes dx^{\alpha_3}\left(\partial_\beta\right) - dx^{\alpha_0} \otimes dx^{\alpha_1} \otimes dx^{\alpha_2}\left(\partial_\beta\right) \otimes dx^{\alpha_3}\left(\partial_\alpha\right))\}$

$= \sum_{\alpha_1\alpha_2\alpha_3\alpha_0} \sum_{\alpha<\beta} Z^{\alpha\beta\alpha\beta} \epsilon\left(\alpha_0, \alpha_1, \alpha_2, \alpha_3\right)\left(\delta^{\alpha_2}_\alpha \delta^{\alpha_3}_\beta dx^{\alpha_0} \otimes dx^{\alpha_1} - \delta^{\alpha_2}_\beta \delta^{\alpha_3}_\alpha dx^{\alpha_0} \otimes dx^{\alpha_1}\right)$

$= \sum_{\alpha_0\alpha_1} \sum_{\alpha<\beta} Z^{\alpha\beta}\left(\epsilon\left(\alpha_0, \alpha_1, \alpha, \beta\right) dx^{\alpha_0} \otimes dx^{\alpha_1} + \epsilon\left(\alpha_0, \alpha_1, \alpha, \beta\right) dx^{\alpha_0} \otimes dx^{\alpha_1}\right)$

$= 2\sum_{\alpha<\beta} Z^{\alpha\beta} \sum_{\alpha_0\alpha_1} \epsilon\left(\alpha_0, \alpha_1, \alpha, \beta\right) dx^{\alpha_0} \otimes dx^{\alpha_1}$

$= 2\sum_{\alpha<\beta} Z^{\alpha\beta}\left(\sum_{\alpha_0<\alpha_1} \epsilon\left(\alpha_0, \alpha_1, \alpha, \beta\right) dx^{\alpha_0} \otimes dx^{\alpha_1} + \sum_{\alpha_0>\alpha_1} \epsilon\left(\alpha_0, \alpha_1, \alpha, \beta\right) dx^{\alpha_0} \otimes dx^{\alpha_1}\right)$

$= 2\sum_{\alpha<\beta} Z^{\alpha\beta}\left(\sum_{\alpha_0<\alpha_1} \epsilon\left(\alpha_0, \alpha_1, \alpha, \beta\right) dx^{\alpha_0} \otimes dx^{\alpha_1} + \sum_{\alpha_1>\alpha_0} \epsilon\left(\alpha_1, \alpha_0, \alpha, \beta\right) dx^{\alpha_1} \otimes dx^{\alpha_0}\right)$

$= 2\sum_{\alpha<\beta} Z^{\alpha\beta} \sum_{\alpha_0<\alpha_1}\left(\epsilon\left(\alpha_0, \alpha_1, \alpha, \beta\right) dx^{\alpha_0} \otimes dx^{\alpha_1} - \epsilon\left(\alpha_0, \alpha_1, \alpha, \beta\right) dx^{\alpha_1} \otimes dx^{\alpha_0}\right)$

$= 2\sum_{\alpha<\beta} Z^{\alpha\beta} \sum_{\alpha_0<\alpha_1} \epsilon\left(\alpha_0, \alpha_1, \alpha, \beta\right)\left(dx^{\alpha_0} \otimes dx^{\alpha_1} - dx^{\alpha_1} \otimes dx^{\alpha_0}\right)$

$\varpi_0(Z) = 2\sum_{\alpha<\beta} Z^{\alpha\beta} \sum_{\alpha_0<\alpha_1} \epsilon(\alpha_0, \alpha_1, \alpha, \beta)\, dx^{\alpha_0} \wedge dx^{\alpha_1}$
$\Pi = \varpi_4 \left(\sum_{\{\alpha\beta\}} Z^{\alpha\beta} \partial_\alpha \wedge \partial_\beta \right)$
$= 2(\det O') \sum_{\lambda<\mu} \sum_{\alpha<\beta,a} Z^{\alpha\beta} \epsilon(\lambda, \mu, \alpha, \beta)\, dx^\lambda \wedge dx^\mu$
Expressed in coordinates :
$\Pi = -2(\det O') \{ Z^{32} dx^0 \wedge dx^1 + Z^{13} dx^{\cdot 0} \wedge dx^2 + Z^{21} dx^0 \wedge dx^3$
$+Z^{03} dx^2 \wedge dx^1 + Z^{02} dx^1 \wedge dx^3 + Z^{01} dx^3 \wedge dx^2 \}$

Notice the choice of indexes : all the formulas are much simpler with this one. Of course it is related to the table 1.

b) Exterior derivative : $d\Pi$
$d\Pi = -2\{\partial_2 (Z^{32} (\det O'))\, dx^2 \wedge dx^0 \wedge dx^1 + \partial_3 (Z^{32} (\det O'))\, dx^3 \wedge dx^0 \wedge dx^1$
$+\partial_1 (Z^{13} (\det O'))\, dx^1 \wedge dx^{\cdot 0} \wedge dx^2 + \partial_3 (Z^{13} (\det O'))\, dx^3 \wedge dx^{\cdot 0} \wedge dx^2$
$+\partial_1 (Z^{21} (\det O'))\, dx^1 \wedge dx^0 \wedge dx^3 + \partial_2 (Z^{21} (\det O'))\, dx^2 \wedge dx^0 \wedge dx^3$
$+\partial_0 (Z^{03} (\det O'))\, dx^0 \wedge dx^2 \wedge dx^1 + \partial_3 (Z^{03} (\det O'))\, dx^3 \wedge dx^2 \wedge dx^1$
$+\partial_0 (Z^{02} (\det O'))\, dx^0 \wedge dx^1 \wedge dx^3 + \partial_2 (Z^{02}_{Ga} (\det O'))\, dx^2 \wedge dx^1 \wedge dx^3$
$+\partial_0 (Z^{01} (\det O'))\, dx^0 \wedge dx^3 \wedge dx^2 + \partial_1 (Z^{01} (\det O'))\, dx^1 \wedge dx^3 \wedge dx^2 \}$
$= -2\{$
$- (\partial_3 (Z^{03} (\det O')) - \partial_2 (Z^{02}_{Ga} (\det O')) - \partial_1 (Z^{01} (\det O')))\, dx^1 \wedge dx^2 \wedge dx^3$
$+ (\partial_3 (Z^{13} (\det O')) + \partial_2 (Z^{12} (\det O')) + \partial_0 (Z^{10} (\det O')))\, dx^0 \wedge dx^2 \wedge dx^3$
$- (\partial_3 (Z^{23} (\det O')) + \partial_1 (Z^{21} (\det O')) + \partial_0 (Z^{20} (\det O')))\, dx^0 \wedge dx^1 \wedge dx^3$
$+ (\partial_2 (Z^{32} (\det O')) + \partial_1 (Z^{31} (\det O')) + \partial_0 (Z^{30} (\det O')))\, dx^0 \wedge dx^1 \wedge dx^2 \}$
$d\Pi = -2\sum_{\alpha=0}^3 \sum_{\beta=0}^3 (-1)^{\alpha+1} \left(\partial_\beta \left(Z^{\alpha\beta} \det O' \right) \right) dx^0 \wedge ... \wedge \widehat{dx^\alpha} \wedge ... \wedge dx^3$
which is conveniently written as :
$d\Pi = d\left(\varpi_4 \left(\sum_{\{\alpha\beta\}} Z^{\alpha\beta} \partial_\alpha \wedge \partial_\beta \right) \right) = -2 \frac{1}{\det O'} \sum_a \varpi_4 \left(\sum_{\alpha\beta} \partial_\beta \left(Z^{\alpha\beta} \det O' \right) \partial_\alpha \right)$

c) Let us compute $\left(\sum_\gamma X_\gamma dx^\gamma \right) \wedge \Pi$ where $X = \sum_\gamma X_\gamma dx^\gamma$ is a one-form:
$\left(\sum_\gamma X_\gamma dx^\gamma \right) \wedge \Pi$
$= -2(\det O') \left(\sum_\gamma X_\gamma dx^\gamma \right) \wedge \{ Z^{32} dx^0 \wedge dx^1 + Z^{13} dx^{\cdot 0} \wedge dx^2$
$+Z^{21} dx^0 \wedge dx^3 + Z^{03} dx^2 \wedge dx^1 + Z^{02} dx^1 \wedge dx^3 + Z^{01} dx^3 \wedge dx^2 \}$
$= -2(\det O') \{$
$- (Z^{03} X_3 + Z^{02} X_2 + Z^{01} X_1)\, dx^1 \wedge dx^2 \wedge dx^3$
$+ (Z^{13} X_3 + Z^{12} X_2 + Z^{10} X_0)\, dx^0 \wedge dx^2 \wedge dx^3$
$- (Z^{23} X_3 + Z^{21} X_1 + Z^{20} X_0)\, dx^0 \wedge dx^1 \wedge dx^3$
$+ (Z^{32} X_2 + Z^{31} X_1 + Z^{30} X_0)\, dx^0 \wedge dx^1 \wedge dx^2 \}$

$= -2\left(\det O'\right)\sum_{\alpha}\left(-1\right)^{\alpha+1}\sum_{\gamma} Z^{\alpha\gamma}X_{\gamma}dx^0\wedge ...\wedge \widehat{dx^{\alpha}}\wedge ...\wedge dx^3$

where the symbol $\hat{}$ over a variable denotes as usual that the variable shall be omitted.

On the other hand :

$\varpi_4\left(\sum_{\alpha\gamma} Z^{\alpha\gamma}X_{\gamma}\partial_{\alpha}\right)$

$= \left(\det O'\right)\sum_{\alpha}\left(-1\right)^{\alpha+1}\sum_{\gamma} Z^{a,\alpha\gamma}X_{\gamma}dx^0\wedge ...\wedge \widehat{dx^{\alpha}}\wedge ...\wedge dx^3$

So one can write :

$\varpi_4\left(\sum_{\alpha\gamma} Z^{\alpha\gamma}X_{\gamma}\partial_{\alpha}\right) = -\frac{1}{2}\left(\sum_{\gamma} X_{\gamma}dx^{\gamma}\right)\wedge\Pi$

$= -\frac{1}{2}\left(\sum_{\gamma} X_{\gamma}dx^{\gamma}\right)\wedge\varpi_4\left(\sum_{\{\alpha\beta\}} Z^{\alpha\beta}\partial_{\alpha}\wedge\partial_{\beta}\right)$

4) From these formulas we have :

$$\Pi_G = \varpi_4\left(\sum_{a,\{\alpha\beta\}}\frac{dL_F}{d\partial_{\alpha}G^a_{\beta}}\partial_{\alpha}\wedge\partial_{\beta}\otimes\overrightarrow{\kappa}_a\right) = 2\sum_{\substack{\alpha<\beta,a\\ \lambda<\mu}}\epsilon\left(\lambda,\mu,\alpha,\beta\right)\frac{d\mathcal{L}_F}{d\partial_{\alpha}G^a_{\beta}}dx^{\lambda}\wedge dx^{\mu}\otimes\overrightarrow{\kappa}_a \tag{57}$$

This is a 2-form on M valued in o(3,1) called superpotential.

$\Pi_G = -2\left(\det O'\right)\sum_a\{Z^{a,32}_G dx^0\wedge dx^1 + Z^{a,13}_G dx^{.0}\wedge dx^2 + Z^{a,21}_G dx^0\wedge dx^3 + Z^{a,03}_G dx^2\wedge dx^1$

$+Z^{a,02}_G dx^1\wedge dx^3 + Z^{a,01}_G dx^3\wedge dx^2\}\otimes\overrightarrow{\kappa}_a$

And the exterior derivative :

$d\Pi_G = -2\frac{1}{\det O'}\sum_a\varpi_4\left(\sum_{\alpha\beta}\partial_{\beta}\left(\frac{d\mathcal{L}}{d\partial_{\alpha}G^a_{\beta}}\right)\partial_{\alpha}\right)\otimes\overrightarrow{\kappa}_a$

$= -2\sum_a\sum_{\alpha\beta=0}^3\left(-1\right)^{\alpha+1}\partial_{\beta}\left(\frac{d\mathcal{L}}{d\partial_{\alpha}G^a_{\beta}}\right)dx^0\wedge ...\wedge\widehat{dx^{\alpha}}\wedge ...\wedge dx^3\otimes\overrightarrow{\kappa}_a$

5) $\frac{dL}{dG^a_{\alpha}} = \sum_{ij} N\left(\frac{dL_M}{d\,\mathrm{Re}\,\nabla_{\alpha}\psi^{ij}}\mathrm{Re}\left(\left[\kappa_a\right]\left[\psi\right]\right)^i_j + \frac{dL_M}{d\,\mathrm{Im}\,\nabla_{\alpha}\psi^{ij}}\mathrm{Im}\left(\left[\kappa_a\right]\left[\psi\right]\right)^i_j\right)$

$+2\sum_{b,\beta}\frac{dL_F}{d\mathcal{F}^b_{G\alpha\beta}}\left[\overrightarrow{\kappa}_a, G_{\beta}\right]^b + \frac{\partial\left(VL^{\diamondsuit}_M + L_F\right)}{\partial G^a_{\alpha}}$

are the components of a tensor field :

$$Y_G = \sum_{a,\alpha}\frac{d\left(NL_M + L_F\right)}{dG^a_{\alpha}}\partial_{\alpha}\otimes\overrightarrow{\kappa}_a \tag{58}$$

Y_G is comprised of one part related to the particles (L_M) and one part related to the gravitational field (L_F).

6) We have :

$\varpi_4 (Y_G)$

$= \sum_a \sum_{\alpha=0}^{3} (-1)^{\alpha+1} \frac{d\mathcal{L}}{dG_\alpha^a} dx^0 \wedge ... \wedge \widehat{dx^\alpha} \wedge ... \wedge dx^3 \otimes \overrightarrow{\kappa}_a$

$= (\det O') \sum_\alpha \sum_{\alpha_1\alpha_2\alpha_3\alpha_4} Y_G^{a\alpha} \epsilon (\alpha_1, \alpha_2, \alpha_3, \alpha_4) dx^{\alpha_1} \otimes dx^{\alpha_2} \otimes dx^{\alpha_3} \otimes dx^{\alpha_4} (\partial_\alpha)$

$= (\det O') \sum_\alpha (-1)^{\alpha+1} Y_G^{a\alpha} dx^0 \wedge ... \wedge \widehat{dx^\alpha} \wedge ... \wedge dx^3$

So the gravitational equation $\forall a, \alpha : 0 = \frac{d\mathcal{L}}{dG_\alpha^a} - \sum_\beta \frac{d}{d\xi^\beta} \left(\frac{d\mathcal{L}}{d\partial_\beta G_\alpha^a} \right)$ is equivalent to :

$\sum_a \sum_{\alpha=0}^{3} (-1)^{\alpha+1} \frac{d\mathcal{L}}{dG_\alpha^a} dx^0 \wedge ... \wedge \widehat{dx^\alpha} \wedge ... \wedge dx^3 \otimes \overrightarrow{\kappa}_a$

$= \sum_a \sum_{\alpha=0}^{3} (-1)^{\alpha+1} \sum_\beta \partial_\beta \left(\frac{d\mathcal{L}}{d\partial_\beta G_\alpha^a} \right) dx^0 \wedge ... \wedge \widehat{dx^\alpha} \wedge ... \wedge dx^3 \otimes \overrightarrow{\kappa}_a$

$\varpi_4 (Y_G) = -\sum_a \sum_{\alpha=0}^{3} (-1)^{\alpha+1} \sum_\beta \partial_\beta \left(\frac{d\mathcal{L}}{d\partial_\alpha G_\beta^a} \right) dx^0 \wedge ... \wedge \widehat{dx^\alpha} \wedge ... \wedge dx^3 \otimes \overrightarrow{\kappa}_a$

$$\varpi_4 (Y_G) = \frac{1}{2} \mathbf{d\Pi}_G \tag{59}$$

As both quantities are tensors, this a fully geometric equation, which does not involve coordinates and can be substituted to the gravitational equation.

7) The integral $\int_{\Omega(t)} d\varpi_4 (Y_G) = \int_{\Omega(t)} \frac{1}{2} d^2 \Pi_G$ over the region delimited by S(0) and S(t) is null, but by Stockes theorem : $\int_{\Omega(t)} d\varpi_4 (Y_G) = 0 = \int_{\partial\Omega(t)} \varpi_4 (Y_G)$. If $Y_G = 0$ on the rim of each S(t) the flux of the vector field Y_G is conserved : $\int_{S(0)} \varpi_4 (Y_G) = \int_{S(t)} \varpi_4 (Y_G)$.

8) With the various gauge constraints :

$Y_G = \sum_{a\alpha} \{ \sum_{ij} \left(\frac{dNL_M}{d \operatorname{Re} \nabla_\alpha \psi^{ij}} \operatorname{Re} ([\kappa_a] [\psi])_j^i + \frac{dNL_M}{d \operatorname{Im} \nabla_\alpha \psi^{ij}} \operatorname{Im} ([\kappa_a] [\psi])_j^i \right)$

$+2 \sum_{b,\beta} \frac{dL_F}{d\mathcal{F}_{G\alpha\beta}^b} [\overrightarrow{\kappa}_a, G_\beta]^b + \frac{\partial L}{\partial G_\alpha^a} \} \partial_\alpha \otimes \overrightarrow{\kappa}_a$

The equation can be written :

$\forall a : \varpi_4 \left(\sum_{\alpha ij} N \left(\frac{dL_M}{d \operatorname{Re} \nabla_\alpha \psi^{ij}} \operatorname{Re} ([\kappa_a] [\psi])_j^i + \frac{dL_M}{d \operatorname{Im} \nabla_\alpha \psi^{ij}} \operatorname{Im} ([\kappa_a] [\psi])_j^i + \frac{\partial L}{\partial G_\alpha^a} \right) \partial_\alpha \right)$

$= \frac{1}{2} d\Pi_G^a - \varpi_4 \left(2 \sum_{b,\beta} \frac{dL_F}{d\mathcal{F}_{G\alpha\beta}^b} [\overrightarrow{\kappa}_a, G_\beta]^b \partial_\alpha \right)$

$\varpi_4 \left(2 \sum_{b,\beta} \frac{dL_F}{d\mathcal{F}_{G\alpha\beta}^b} [\overrightarrow{\kappa}_a, G_\beta]^b \partial_\alpha \right) = \sum_b \varpi_4 \left(\sum_\beta Z_G^{b,\alpha\beta} [\overrightarrow{\kappa}_a, G_\beta]^b \partial_\alpha \right)$

$= -\frac{1}{2} \sum_b \left(\sum_\beta [\overrightarrow{\kappa}_a, G_\beta]^b dx^\beta \right) \wedge \Pi_G^b$

So the conservation equation reads :

$\forall a :$

$\varpi_4 \left(\sum_{\alpha ij} N \left(\frac{dL_M}{d\operatorname{Re}\nabla_\alpha \psi^{ij}} \operatorname{Re}\left([\kappa_a][\psi]\right)^i_j + \frac{dL_M}{d\operatorname{Im}\nabla_\alpha \psi^{ij}} \operatorname{Im}\left([\kappa_a][\psi]\right)^i_j\right)\partial_\alpha\right) + \varpi_4 \left(\frac{\partial L}{\partial G^a_\alpha}\partial_\alpha\right)$
$= \frac{1}{2} d\Pi^a_G - \sum_b \left(\sum_\beta \left[\overrightarrow{\kappa}_a, G_\beta\right]^b dx^\beta\right) \wedge \Pi^b_G$

9.3 Noether currents for the other fields

We will proceed in the same way as above, with just one complication coming from the complex value of the quantities.

1) Let us fix $\overrightarrow{\theta} = \sum_a \theta^a \overrightarrow{\theta}_a = Ct$ then the vector Y has the components :
$Y^{\operatorname{Re}\psi^{ij}} = \sum_a \operatorname{Re}\theta^a \operatorname{Re}\left(\left[\psi^\diamondsuit\right][\theta_a]^t\right)^i_j - \operatorname{Im}\theta^a \operatorname{Im}\left(\left[\psi^\diamondsuit\right][\theta_a]^t\right)^i_j$
$Y^{\operatorname{Im}\psi^{ij}} = \sum_a \operatorname{Re}\theta^a \operatorname{Im}\left(\left[\psi^\diamondsuit\right][\theta_a]^t\right)^i_j + \operatorname{Im}\theta^a \operatorname{Re}\left(\left[\psi^\diamondsuit\right][\theta_a]^t\right)^i_j$
$Y^{\operatorname{Re}\grave{A}^a_\beta} = \sum_{bc} C^a_{bc}\left(\operatorname{Re}\theta^b \operatorname{Re}\acute{A}^c_\beta - \operatorname{Im}\theta^b \operatorname{Im}\acute{A}^c_\beta\right)$
$Y^{\operatorname{Im}\grave{A}^a_\beta} = \sum_{bc} C^a_{bc}\left(\operatorname{Re}\theta^b \operatorname{Im}\acute{A}^c_\beta + \operatorname{Im}\theta^b \operatorname{Re}\acute{A}^c_\beta\right)$
And :
$\frac{d\mathcal{L}}{d\operatorname{Re}\partial_\alpha\psi^{ij}} = N\frac{dL_M \det O'}{d\operatorname{Re}\nabla_\alpha\psi^{ij}}; \frac{d\mathcal{L}}{d\operatorname{Im}\partial_\alpha\psi^{ij}} = N\frac{dL_M \det O'}{d\operatorname{Im}\nabla_\alpha\psi^{ij}}$
$\frac{d\mathcal{L}}{d\operatorname{Re}\partial_\alpha\grave{A}^a_\beta} = 2\frac{dL_F}{d\operatorname{Re}\mathcal{F}^b_{A,\alpha\beta}}; \frac{d\mathcal{L}}{d\operatorname{Im}\partial_\alpha\grave{A}^a_\beta} = 2\frac{dL_F}{d\operatorname{Im}\mathcal{F}^b_{A,\alpha\beta}}$
$\left(\overrightarrow{\theta}_a\right)$ is a real basis of the complexified $T_1U^c = T_1U \oplus iT_1U$ with complex components. The set $\left(\overrightarrow{\theta}_a, i\overrightarrow{\theta}_a\right)$ is a real basis of the real vector space T_1U^c with the real components $\operatorname{Re}\theta^a, \operatorname{Im}\theta^a$.

So we have to consider two quantities :
$\sum_j \frac{\partial\mathcal{L}}{\partial z^j_\alpha} Y^j\left(\operatorname{Re}\overrightarrow{\theta}\right)$
$= \sum_a \{\sum_{ij} \frac{d\mathcal{L}}{d\operatorname{Re}\partial_\alpha\psi^{ij}} \operatorname{Re}\theta^a \operatorname{Re}\left(\left[\psi^\diamondsuit\right][\theta_a]^t\right)^i_j + \frac{d\mathcal{L}}{d\operatorname{Im}\partial_\alpha\psi^{ij}} \operatorname{Re}\theta^a \operatorname{Im}\left(\left[\psi^\diamondsuit\right][\theta_a]^t\right)^i_j$
$+ \sum_{b\beta} \frac{d\mathcal{L}}{d\operatorname{Re}\partial_\alpha\grave{A}^a_\beta} C^a_{bc} \operatorname{Re}\theta^b \operatorname{Re}\grave{A}^c_\beta + \frac{d\mathcal{L}}{d\operatorname{Im}\partial_\alpha\grave{A}^a_\beta} C^a_{bc} \operatorname{Re}\theta^b \operatorname{Im}\grave{A}^c_\beta\}$
$= \sum_a (\operatorname{Re}\theta^a)\{\sum_{ij} N\frac{dL_M \det O'}{d\operatorname{Re}\nabla_\alpha\psi^{ij}} \operatorname{Re}\left(\left[\psi^\diamondsuit\right][\theta_a]^t\right)^i_j + N\frac{dL_M \det O'}{d\operatorname{Im}\nabla_\alpha\psi^{ij}} \operatorname{Im}\left(\left[\psi^\diamondsuit\right][\theta_a]^t\right)^i_j$
$+2\sum_{b\beta} \frac{dL_F \det O'}{d\operatorname{Re}\mathcal{F}^b_{A,\alpha\beta}} \operatorname{Re}\left[\overrightarrow{\theta}_a, \grave{A}_\beta\right]^b + \frac{dL_F \det O'}{d\operatorname{Im}\mathcal{F}^b_{A,\alpha\beta}} \operatorname{Im}\left[\overrightarrow{\theta}_a, \grave{A}_\beta\right]^b\}$
and
$\sum_j \frac{\partial\mathcal{L}}{\partial z^j_\alpha} Y^j\left(\operatorname{Im}\overrightarrow{\theta}\right)$
$= \sum_a \{\sum_{ij} -\frac{d\mathcal{L}}{d\operatorname{Re}\partial_\alpha\psi^{ij}} \operatorname{Im}\theta^a \operatorname{Im}\left(\left[\psi^\diamondsuit\right][\theta_a]^t\right)^i_j + \frac{d\mathcal{L}}{d\operatorname{Im}\partial_\alpha\psi^{ij}} \operatorname{Im}\theta^a \operatorname{Re}\left(\left[\psi^\diamondsuit\right][\theta_a]^t\right)^i_j$
$+ \sum_{b\beta} -\frac{d\mathcal{L}}{d\operatorname{Re}\partial_\alpha\grave{A}^a_\beta} C^a_{bc} \operatorname{Im}\theta^b \operatorname{Im}\grave{A}^c_\beta + \frac{d\mathcal{L}}{d\operatorname{Im}\partial_\alpha\grave{A}^a_\beta} C^a_{bc} \operatorname{Im}\theta^b \operatorname{Re}\grave{A}^c_\beta\}$
$= \sum_a (\operatorname{Im}\theta^a)\{N\sum_{ij} -\frac{dL_M \det O'}{d\operatorname{Re}\nabla_\alpha\psi^{ij}} \operatorname{Im}\left(\left[\psi^\diamondsuit\right][\theta_a]^t\right)^i_j + \frac{dL_M \det O'}{d\operatorname{Im}\nabla_\alpha\psi^{ij}} \operatorname{Re}\left(\left[\psi^\diamondsuit\right][\theta_a]^t\right)^i_j$

$+2\sum_{b\beta} -\frac{dL_F \det O'}{d\operatorname{Re}\mathcal{F}^b_{A,\alpha\beta}} \operatorname{Im}\left[\overrightarrow{\theta}_a, \grave{A}_\beta\right]^b + \frac{dL_F \det O'}{d\operatorname{Im}\mathcal{F}^b_{A,\alpha\beta}} \operatorname{Re}\left[\overrightarrow{\theta}_a, \grave{A}_\beta\right]^b \}$

That is, with the gauge invariance identities :

$\sum_j \frac{\partial\mathcal{L}}{\partial z^j_\alpha} Y^j \left(\operatorname{Re}\overrightarrow{\theta}\right) = \sum_a \left(\operatorname{Re}\theta^a\right) \frac{d\mathcal{L}}{d\operatorname{Re}\grave{A}^a_\alpha}$

$\sum_j \frac{\partial\mathcal{L}}{\partial z^j_\alpha} Y^j \left(\operatorname{Im}\overrightarrow{\theta}\right) = \sum_a \left(\operatorname{Im}\theta^a\right) \frac{d\mathcal{L}}{d\operatorname{Im}\grave{A}^a_\alpha}$

The equations 53,54 can be written as :

$\forall a, \alpha : \frac{\delta S}{\delta \operatorname{Re}\grave{A}^a_\alpha} = \frac{d\mathcal{L}}{d\operatorname{Re}\grave{A}^a_\alpha} - \sum_\beta \frac{d}{d\xi^\beta}\left(\frac{d\mathcal{L}}{d\partial_\beta \operatorname{Re}\grave{A}^a_\alpha}\right) = 0$

$\forall a, \alpha : \frac{\delta S}{\delta \operatorname{Im}\grave{A}^a_\alpha} = \frac{d\mathcal{L}}{d\operatorname{Im}\grave{A}^a_\alpha} - \sum_\beta \frac{d}{d\xi^\beta}\left(\frac{d\mathcal{L}}{d\partial_\beta \operatorname{Im}\grave{A}^a_\alpha}\right) = 0$

So :

$\sum_j \frac{\partial\mathcal{L}}{\partial z^j_\alpha} Y^j \left(\operatorname{Re}\overrightarrow{\theta}\right) = \sum_a \left(\operatorname{Re}\theta^a\right)\left(\frac{\delta S}{\delta \operatorname{Re}\grave{A}^a_\alpha} + \sum_\beta \frac{d}{d\xi^\beta}\left(\frac{d\mathcal{L}}{d\partial_\beta \operatorname{Re}\grave{A}^a_\alpha}\right)\right)$

$\sum_j \frac{\partial\mathcal{L}}{\partial z^j_\alpha} Y^j \left(\operatorname{Im}\overrightarrow{\theta}\right) = \sum_a \left(\operatorname{Im}\theta^a\right)\left(\frac{\delta S}{\delta \operatorname{Im}\grave{A}^a_\alpha} + \sum_\beta \frac{d}{d\xi^\beta}\left(\frac{d\mathcal{L}}{d\partial_\beta \operatorname{Im}\grave{A}^a_\alpha}\right)\right)$

and on shell : $\forall \overrightarrow{\theta} : \sum_\alpha \frac{d}{d\xi^\alpha}\left(\sum_i \frac{\partial\mathcal{L}}{\partial z^i_\alpha} Y^i \left(\operatorname{Re}\overrightarrow{\theta}\right)\right) = \sum_a \left(\operatorname{Re}\theta^a\right) \sum_{\alpha\beta} \frac{d}{d\xi^\alpha}\frac{d}{d\xi^\beta}\left(\frac{d\mathcal{L}}{d\partial_\beta \operatorname{Re}\grave{A}^a_\alpha}\right) =$ 0 because $\frac{d\mathcal{L}}{d\partial_\beta \operatorname{Re}\grave{A}^a_\alpha} + \frac{d\mathcal{L}}{d\partial_\alpha \operatorname{Re}\grave{A}^a_\beta} = 0$

2) The quantities : $\frac{d\mathcal{L}}{d\operatorname{Re}\partial_\alpha \grave{A}^a_\beta} = 2\frac{dL_F}{d\operatorname{Re}\mathcal{F}^a_{A,\alpha\beta}}, \frac{d\mathcal{L}}{d\operatorname{Im}\partial_\alpha \grave{A}^a_\beta} = 2\frac{dL_F}{d\operatorname{Im}\mathcal{F}^a_{A,\alpha\beta}}$ are the components of the anti-symmetric 2-vector:

$Z^a_{AR} = \sum_{\{\alpha\beta\}} \frac{d\mathcal{L}}{d\operatorname{Re}\partial_\alpha \grave{A}^a_\beta} \partial_\alpha \wedge \partial_\beta, Z^a_{AI} = \sum_{\{\alpha\beta\}} \frac{d\mathcal{L}}{d\operatorname{Im}\partial_\alpha \grave{A}^a_\beta} \partial_\alpha \wedge \partial_\beta.$

The 2-forms $\Pi^a_{AR} = \varpi_4\left(Z^a_{AR}\right), \Pi^a_{AI} = \varpi_4\left(Z^a_{AI}\right)$ can be computed as above :

$$\Pi^a_{AR} = \varpi_4\left(Z^a_{AR}\right) = 2\sum_{\lambda<\mu}\sum_{\alpha<\beta,a} \epsilon\left(\lambda,\mu,\alpha,\beta\right) \frac{d\mathcal{L}}{d\operatorname{Re}\partial_\alpha \grave{A}^a_\beta} dx^\lambda \wedge dx^\mu \tag{60}$$

$$\Pi^a_{AI} = \varpi_4\left(Z^a_{AR}\right) = 2\sum_{\lambda<\mu}\sum_{\alpha<\beta,a} \epsilon\left(\lambda,\mu,\alpha,\beta\right) \frac{d\mathcal{L}}{d\operatorname{Im}\partial_\alpha \grave{A}^a_\beta} dx^\lambda \wedge dx^\mu \tag{61}$$

They are the superpotentiels of the field À. Their exterior derivative are :

$d\Pi^a_{AR} = -2\sum_{\alpha=0}^3 \left(\sum_{\beta=0}^3 \left(-1\right)^{\alpha+1} \partial_\beta\left(\frac{d\mathcal{L}}{d\operatorname{Re}\partial_\alpha \grave{A}^a_\beta}\right)\right) dx^0 \wedge ... \wedge \widehat{dx^\alpha} \wedge ... \wedge dx^3$

$= -2\frac{1}{\det O'} \sum_a \varpi_4\left(\sum_{\alpha\beta} \partial_\beta\left(\frac{d\mathcal{L}}{d\operatorname{Re}\partial_\alpha \grave{A}^a_\beta}\right)\partial_\alpha\right)$

$d\Pi_{AI}^{a}=-2\sum_{\alpha=0}^{3}\left(\sum_{\beta=0}^{3}(-1)^{\alpha+1}\partial_{\beta}\left(\frac{d\mathcal{L}}{d\operatorname{Im}\partial_{\alpha}\grave{A}_{\beta}^{a}}\right)\right)dx^{0}\wedge...\wedge\widehat{dx^{\alpha}}\wedge...\wedge dx^{3}$
$=-2\frac{1}{\det O'}\sum_{a}\varpi_{4}\left(\sum_{\alpha\beta}\partial_{\beta}\left(\frac{d\mathcal{L}}{d\operatorname{Im}\partial_{\alpha}\grave{A}_{\beta}^{a}}\right)\partial_{\alpha}\right)$

3) The quantities :$Y_{AR}^{a,\alpha}=\frac{d(NL_M+L_F)}{d\operatorname{Re}\grave{A}_{\alpha}^{a}}, Y_{AI}^{a,\alpha}=\frac{d(VL_M+L_F)}{d\operatorname{Im}\grave{A}_{\alpha}^{a}}$ are the components of vector fields Y_{AR}^{a},Y_{AI}^{a} :

$$\mathbf{Y}_{AR}^{a}=\sum_{\alpha}\frac{d\left(NL_{M}+L_{F}\right)}{d\operatorname{Re}\grave{A}_{\alpha}^{a}}\}\partial_{\alpha};\mathbf{Y}_{AI}^{a}=\sum_{\alpha}\frac{d\left(VL_{M}+L_{F}\right)}{d\operatorname{Im}\grave{A}_{\alpha}^{a}}\}\partial_{\alpha} \qquad (62)$$

and :
$\varpi_{4}\left(Y_{AR}^{a}\right)=\sum_{\alpha=0}^{3}(-1)^{\alpha+1}\left(Y_{AR}^{a,\alpha}\right)(\det O')\,dx^{0}\wedge...\wedge\widehat{dx^{\alpha}}\wedge...\wedge dx^{3}$
$\varpi_{4}\left(Y_{AI}^{a}\right)=\sum_{\alpha=0}^{3}(-1)^{\alpha+1}\left(Y_{AI}^{a,\alpha}\right)(\det O')\,dx^{0}\wedge...\wedge\widehat{dx^{\alpha}}\wedge...\wedge dx^{3}$
The equations 53,54 are equivalent to :

$$\varpi_{4}\left(Y_{AR}^{a}\right)=\frac{1}{2}\mathbf{d\Pi}_{AR}^{a};\varpi_{4}\left(Y_{AI}^{a}\right)=\frac{1}{2}\mathbf{d\Pi}_{AI}^{a} \qquad (63)$$

As seen previously the flow of the 2m vectors field Y_{AR}^{a},Y_{AI}^{a} is conserved.

4) We can proceed to calculations similar as above :
$Y_{AR}^{\alpha a}=N\sum_{ij}\frac{dL_{M}}{d\operatorname{Re}\nabla_{\alpha}\psi^{ij}}\operatorname{Re}\left(\left[\psi^{\diamond}\right]\left[\theta_{a}\right]^{t}\right)_{j}^{i}+\frac{dL_{M}}{d\operatorname{Im}\nabla_{\alpha}\psi^{ij}}\operatorname{Im}\left(\left[\psi^{\diamond}\right]\left[\theta_{a}\right]^{t}\right)_{j}^{i}$
$+2\sum_{b\beta}\frac{dL_{F}}{d\operatorname{Re}\mathcal{F}_{A,\alpha\beta}^{b}}\operatorname{Re}\left[\overrightarrow{\theta}_{a},\grave{A}_{\beta}\right]^{b}+\frac{dL_{F}}{d\operatorname{Im}\mathcal{F}_{A,\alpha\beta}^{b}}\operatorname{Im}\left[\overrightarrow{\theta}_{a},\grave{A}_{\beta}\right]^{b}$
$Y_{AI}^{\alpha a}=N\sum_{ij}-\frac{dL_{M}}{d\operatorname{Re}\nabla_{\alpha}\psi^{ij}}\operatorname{Im}\left(\left[\psi^{\diamond}\right]\left[\theta_{a}\right]^{t}\right)_{j}^{i}+\frac{dL_{M}}{d\operatorname{Im}\nabla_{\alpha}\psi^{ij}}\operatorname{Re}\left(\left[\psi^{\diamond}\right]\left[\theta_{a}\right]^{t}\right)_{j}^{i}$
$+2\sum_{b\beta}-\frac{dL_{F}}{d\operatorname{Re}\mathcal{F}_{A,\alpha\beta}^{b}}\operatorname{Im}\left[\overrightarrow{\theta}_{a},\grave{A}_{\beta}\right]^{b}+\frac{dL_{F}}{d\operatorname{Im}\mathcal{F}_{A,\alpha\beta}^{b}}\operatorname{Re}\left[\overrightarrow{\theta}_{a},\grave{A}_{\beta}\right]^{b}$
$\varpi_{4}\left(2\sum_{\beta}\frac{dL_{F}}{d\operatorname{Re}\mathcal{F}_{A\alpha\beta}^{b}}\operatorname{Re}\left[\overrightarrow{\theta}_{a},\grave{A}_{\beta}\right]^{b}\partial_{\alpha}\right)=\varpi_{4}\left(\sum_{\alpha\beta}Z_{AR}^{b\alpha\beta}\operatorname{Re}\left[\overrightarrow{\theta}_{a},\grave{A}_{\beta}\right]^{b}\partial_{\alpha}\right)$
$=-\frac{1}{2}\left(\sum_{\beta}\operatorname{Re}\left[\overrightarrow{\theta}_{a},\grave{A}_{\beta}\right]^{b}dx^{\beta}\right)\wedge\Pi_{AR}^{b}=-\frac{1}{2}\sum_{bc}C_{ac}^{b}\left(\operatorname{Re}\grave{A}^{c}\right)\wedge\Pi_{AR}^{b}$
$\varpi_{4}\left(2\sum_{\beta}\frac{dL_{F}}{d\operatorname{Im}\mathcal{F}_{A\alpha\beta}^{b}}\operatorname{Im}\left[\overrightarrow{\theta}_{a},\grave{A}_{\beta}\right]^{b}\partial_{\alpha}\right)=-\frac{1}{2}\left(\sum_{\beta}\operatorname{Im}\left[\overrightarrow{\theta}_{a},\grave{A}_{\beta}\right]^{b}dx^{\beta}\right)\wedge\Pi_{AI}^{b}$
$=-\frac{1}{2}\sum_{bc}C_{ac}^{b}\left(\operatorname{Im}\grave{A}^{c}\right)\wedge\Pi_{AI}^{b}$

$\varpi_4 \left(2\sum_\beta \frac{dL_F}{d\operatorname{Re}\mathcal{F}^b_{A\alpha\beta}} \operatorname{Im}\left[\overrightarrow{\theta}_a, \grave{A}_\beta\right]^b \partial_\alpha \right) = -\frac{1}{2}\left(\sum_\beta \operatorname{Im}\left[\overrightarrow{\theta}_a, \grave{A}_\beta\right]^b dx^\beta\right) \wedge \Pi^b_{AR}$
$= -\frac{1}{2}\sum_{bc} C^b_{ac}\left(\operatorname{Im}\grave{A}^c\right) \wedge \Pi^b_{AR}$
$\varpi_4 \left(2\sum_\beta \frac{dL_F}{d\operatorname{Im}\mathcal{F}^b_{A\alpha\beta}} \operatorname{Re}\left[\overrightarrow{\theta}_a, \grave{A}_\beta\right]^b \partial_\alpha \right) = -\frac{1}{2}\left(\sum_\beta \operatorname{Re}\left[\overrightarrow{\theta}_a, \grave{A}_\beta\right]^b dx^\beta\right) \wedge \Pi^b_{AI}$
$= -\frac{1}{2}\sum_{bc} C^b_{ac}\left(\operatorname{Re}\grave{A}^c\right) \wedge \Pi^b_{AI}$
Thus the conservation equations read :
$\varpi_4 \left(N\sum_{ij}\left(\frac{dL_M}{d\operatorname{Re}\nabla_\alpha\psi^{ij}} \operatorname{Re}\left(\left[\psi^\diamond\right]\left[\theta_a\right]^t\right)^i_j + \frac{dL_M}{d\operatorname{Im}\nabla_\alpha\psi^{ij}} \operatorname{Im}\left(\left[\psi^\diamond\right]\left[\theta_a\right]^t\right)^i_j \right)\partial_\alpha\right)$
$= \frac{1}{2}\left(d\Pi^a_{AR} + \sum_{bc} C^b_{ac}\left(\operatorname{Re}\grave{A}^c \wedge \Pi^b_{AR} + \operatorname{Im}\grave{A}^c \wedge \Pi^b_{AI}\right)\right)$
$\varpi_4 \left(N\sum_{ij}\left(-\frac{dL_M}{d\operatorname{Re}\nabla_\alpha\psi^{ij}} \operatorname{Im}\left(\left[\psi^\diamond\right]\left[\theta_a\right]^t\right)^i_j + \frac{dL_M}{d\operatorname{Im}\nabla_\alpha\psi^{ij}} \operatorname{Re}\left(\left[\psi^\diamond\right]\left[\theta_a\right]^t\right)^i_j \right)\partial_\alpha\right)$
$= \frac{1}{2}\left(d\Pi^a_{AR} + \sum_{bc} C^b_{ac}\left(-\operatorname{Im}\grave{A}^c \wedge \Pi^b_{AR} + \operatorname{Re}\grave{A}^c \wedge \Pi^b_{AI}\right)\right)$

5) Remark : obviously we could combine both the real and the imaginary part, that we will do later on, but so far it does not make the computations simpler.

10 THE ENERGY MOMENTUM TENSOR

There are several ways to introduce the energy-momentum tensor. Because the lagrangian does not depend explicitly of m (for covariance reason) the Lagrange equations admit a first integral which is a conserved quantity. One can also look for one parameter groups of diffeomorphisms over the cotangent bundle in a way similar at what we have done for the gauge equivariance. But here a more direct approach is simpler. In a first step we will prove conservation laws of the kind encountered before, involving a "Noether-like" current $Y_{H\beta}$ and a super-potential. But from there it is possible to prove a much stronger result, that we can call "super-conservation laws".

10.1 Noether-like current

1) Let $\widetilde{Y}^\alpha_{H\beta}$ be the quantities : $\widetilde{Y}^\alpha_{H\beta} = \sum_{i>0} \frac{\partial L}{\partial z^i_\alpha}\partial_\beta z^i - \delta^\alpha_\beta L$ (without V) where $L = NL_M + L_F$. It is easily checked that they are the components of a tensor. Indeed in a change of charts (see "covariance") :

$\widehat{\widetilde{Y}_{H\beta}^{\alpha}} = \sum_{\lambda\mu} \left(\sum_i K_\lambda^\alpha \frac{\partial L}{\partial z_\alpha^i} J_\beta^\mu \partial_\mu z^i - K_\lambda^\alpha J_\beta^\mu \delta_\mu^\lambda L \right) = \sum_{\lambda\mu} K_\lambda^\alpha J_\beta^\mu \widetilde{Y}_{H\mu}^{\gamma}$
$\widetilde{Y}_H = \sum_{\alpha\beta} \widetilde{Y}_{H\beta}^{\alpha} dx^\beta \otimes \partial_\alpha$
So *for* β *fixed* we can consider the vector field
$Y_{H\beta} = \widetilde{Y}_{H\beta}^{\alpha} \partial_\alpha - \frac{dNL_M}{dV^\beta} V^\alpha \partial_\alpha = \sum_\alpha \left(-\frac{dNL_M}{dV^\beta} V^\alpha + \sum_{i>1} \frac{\partial L}{\partial z_\alpha^i} \partial_\beta z^i - \delta_\beta^\alpha L \right) \partial_\alpha.$
Its value is :

$$Y_{H\beta}^{\alpha} = -\frac{dNL_M}{dV^\beta} V^\alpha + \sum_{ij} \frac{dNL_M}{d \operatorname{Re} \partial_\alpha \psi^{ij}} \operatorname{Re} \partial_\beta \psi^{ij} + \frac{dNL_M}{d \operatorname{Im} \partial_\alpha \psi^{ij}} \operatorname{Im} \partial_\beta \psi^{ij}$$
$$+ \sum_{a,\gamma} \frac{dL_F}{d\partial_\alpha G_\gamma^a} \partial_\beta G_\gamma^a + \frac{dL_F}{d \operatorname{Re} \partial_\alpha \grave{A}_\gamma^a} \operatorname{Re} \partial_\beta \grave{A}_\gamma^a + \frac{dL_F}{d \operatorname{Im} \partial_\alpha \grave{A}_\gamma^a} \operatorname{Im} \partial_\beta \grave{A}_\gamma^a$$
$$+ \sum_{i\gamma} \frac{dL}{d\partial_\alpha O_\gamma'^i} \partial_\beta O_\gamma'^i - \delta_\beta^\alpha L \tag{64}$$

The index β plays for the "Noether-like" current $Y_{H\beta}$ a role similar to the indexes "a" in the other Noether currents.

2) Using the covariance identities 43,44 we have :
$\forall \alpha, \beta : -\frac{dNL_M}{dV^\beta} V^\alpha + N \sum_{i,j} \frac{dL_M}{d \operatorname{Re} \partial_\alpha \psi^{ij}} \operatorname{Re} \partial_\beta \psi^{ij} + \frac{dL_M}{d \operatorname{Im} \partial_\alpha \psi^{ij}} \operatorname{Im} \partial_\beta \psi^{ij}$
$= -N(\sum_a \frac{dL_M}{dG_\alpha^a} G_\beta^a + \frac{dL_M}{d \operatorname{Re} \grave{A}_\alpha^a} \operatorname{Re} \grave{A}_\beta^a + \frac{dL_M}{d \operatorname{Im} \grave{A}_\alpha^a} \operatorname{Im} \grave{A}_\beta^a$
$+ \sum_i \left(\frac{dL_M}{dO_\alpha'^i} O_\beta'^i + \sum_\lambda \frac{dL_M}{d\partial_\lambda O_\alpha'^i} \partial_\lambda O_\beta'^i + \frac{dL_M}{d\partial_\alpha O_\lambda'^i} \partial_\beta O_\lambda'^i \right))$
$\forall \alpha, \beta : \sum_{a,\gamma} \left(\frac{dL_F}{d \operatorname{Re} \partial_\alpha \grave{A}_\gamma^a} \operatorname{Re} \left(\partial_\beta \grave{A}_\gamma^a \right) + \frac{dL_F}{d \operatorname{Im} \partial_\alpha \grave{A}_\gamma^a} \operatorname{Im} \left(\partial_\beta \grave{A}_\gamma^a \right) + \frac{dL_F}{d\partial_\alpha G_\gamma^a} \left(\partial_\beta G_\gamma^a \right) \right)$
$= \sum_{a,\gamma} \left(\frac{dL_F}{d \operatorname{Re} \partial_\alpha \grave{A}_\gamma^a} \operatorname{Re} \left(\partial_\gamma \grave{A}_\beta^a \right) + \frac{dL_F}{d \operatorname{Im} \partial_\alpha \grave{A}_\gamma^a} \operatorname{Im} \partial_\gamma \grave{A}_\beta^a + \frac{dL_F}{d\partial_\alpha G_\gamma^a} \partial_\gamma G_\beta^a \right)$
$- \sum_a \left(\frac{dL_F}{dG_\alpha^a} G_\beta^a + \frac{dL_F}{d \operatorname{Re} \grave{A}_\alpha^a} \operatorname{Re} \grave{A}_\beta^a + \frac{dL_F}{d \operatorname{Im} \grave{A}_\alpha^a} \operatorname{Im} \grave{A}_\beta^a \right)$
$- \sum_i \left(\frac{dL_F}{dO_\alpha'^i} O_\beta'^i + \sum_\lambda \frac{dL_F}{d\partial_\lambda O_\alpha'^i} \partial_\lambda O_\beta'^i + \frac{dL_F}{d\partial_\alpha O_\lambda'^i} \partial_\beta O_\lambda'^i \right)$
So :
$Y_{H\beta}^{\alpha} = -\sum_a \left(\frac{dNL_M}{dG_\alpha^a} G_\beta^a + \frac{dNL_M}{d \operatorname{Re} \grave{A}_\alpha^a} \operatorname{Re} \grave{A}_\beta^a + \frac{dNL_M}{d \operatorname{Im} \grave{A}_\alpha^a} \operatorname{Im} \grave{A}_\beta^a \right)$
$- \sum_i \left(\frac{dNL_M}{dO_\alpha'^i} O_\beta'^i + \sum_\lambda \frac{dNL_M}{d\partial_\lambda O_\alpha'^i} \partial_\lambda O_\beta'^i + \frac{dNL_M}{d\partial_\alpha O_\lambda'^i} \partial_\beta O_\lambda'^i \right)$
$+ \sum_{a,\gamma} \left(\frac{dL_F}{d \operatorname{Re} \partial_\alpha \grave{A}_\gamma^a} \operatorname{Re} \left(\partial_\gamma \grave{A}_\beta^a \right) + \frac{dL_F}{d \operatorname{Im} \partial_\alpha \grave{A}_\gamma^a} \operatorname{Im} \partial_\gamma \grave{A}_\beta^a + \frac{dL_F}{d\partial_\alpha G_\gamma^a} \partial_\gamma G_\beta^a \right)$
$- \sum_a \left(\frac{dL_F}{dG_\alpha^a} G_\beta^a + \frac{dL_F}{d \operatorname{Re} \grave{A}_\alpha^a} \operatorname{Re} \grave{A}_\beta^a + \frac{dL_F}{d \operatorname{Im} \grave{A}_\alpha^a} \operatorname{Im} \grave{A}_\beta^a \right)$
$- \sum_i \left(\frac{dL_F}{dO_\alpha'^i} O_\beta'^i + \sum_\lambda \frac{dL_F}{d\partial_\lambda O_\alpha'^i} \partial_\lambda O_\beta'^i + \frac{dL_F}{d\partial_\alpha O_\lambda'^i} \partial_\beta O_\lambda'^i \right) + \sum_{i\gamma} \frac{dL}{d\partial_\alpha O_\gamma'^i} \partial_\beta O_\gamma'^i - \delta_\beta^\alpha L$

$= -\sum_a \left(\frac{dL}{dG^a_\alpha}G^a_\beta + \frac{dL}{d\operatorname{Re}\grave{A}^a_\alpha}\operatorname{Re}\grave{A}^a_\beta + \frac{dL}{d\operatorname{Im}\grave{A}^a_\alpha}\operatorname{Im}\grave{A}^a_\beta\right)$
$+\sum_{a,\gamma}\left(\frac{dL}{d\operatorname{Re}\partial_\alpha\grave{A}^a_\gamma}\operatorname{Re}\partial_\gamma\grave{A}^a_\beta + \frac{dL}{d\operatorname{Im}\partial_\alpha\grave{A}^a_\gamma}\operatorname{Im}\partial_\gamma\grave{A}^a_\beta + \frac{dL}{d\partial_\alpha G^a_\gamma}\partial_\gamma G^a_\beta\right)$
$-\sum_i\left(\frac{dL}{dO'^i_\alpha}O'^i_\beta + \sum_\gamma \frac{dL}{d\partial_\gamma O^{i\prime}_\alpha}\partial_\gamma O'^i_\beta\right) - \delta^\alpha_\beta L$
$Y^\alpha_{H\beta}\det O'$
$= -\sum_a \left(\frac{d\mathcal{L}}{dG^a_\alpha}G^a_\beta + \frac{d\mathcal{L}}{d\operatorname{Re}\grave{A}^a_\alpha}\operatorname{Re}\grave{A}^a_\beta + \frac{d\mathcal{L}}{d\operatorname{Im}\grave{A}^a_\alpha}\operatorname{Im}\grave{A}^a_\beta\right)$
$+\sum_{a,\gamma}\left(\frac{d\mathcal{L}}{d\operatorname{Re}\partial_\alpha\grave{A}^a_\gamma}\operatorname{Re}\partial_\gamma\grave{A}^a_\beta + \frac{d\mathcal{L}}{d\operatorname{Im}\partial_\alpha\grave{A}^a_\gamma}\operatorname{Im}\partial_\gamma\grave{A}^a_\beta + \frac{d\mathcal{L}}{d\partial_\alpha G^a_\gamma}\partial_\gamma G^a_\beta\right)$
$-\sum_i\left(\frac{d\mathcal{L}}{dO'^i_\alpha}O'^i_\beta + \sum_\gamma \frac{d\mathcal{L}}{d\partial_\gamma O^{i\prime}_\alpha}\partial_\gamma O'^i_\beta\right)$
where we used :
$\sum_i \frac{dL}{dO'^i_\alpha}O'^i_\beta \det O' = \sum_i \frac{d\mathcal{L}}{dO'^i_\alpha}O'^i_\beta - \sum_i \frac{d\det O'}{dO'^i_\alpha}LO'^i_\beta$
$= \sum_i \frac{d\mathcal{L}}{dO'^i_\alpha}O'^i_\beta - \sum_i O^\alpha_i\left(\det O'\right)LO'^i_\beta = \sum_i \frac{d\mathcal{L}}{dO'^i_\alpha}O'^i_\beta - \delta^\alpha_\beta\mathcal{L}$
$Y^\alpha_{H\beta}\det O'$
$= -\sum_a \left(\frac{d\mathcal{L}}{dG^a_\alpha}G^a_\beta + \frac{d\mathcal{L}}{d\operatorname{Re}\grave{A}^a_\alpha}\operatorname{Re}\grave{A}^a_\beta + \frac{d\mathcal{L}}{d\operatorname{Im}\grave{A}^a_\alpha}\operatorname{Im}\grave{A}^a_\beta\right)$
$+\sum_{a,\gamma}\{\frac{d}{d\xi^\gamma}\left(\frac{d\mathcal{L}}{d\operatorname{Re}\partial_\alpha\grave{A}^a_\gamma}\operatorname{Re}\grave{A}^a_\beta\right) - \left(\frac{d}{d\xi^\gamma}\frac{d\mathcal{L}}{d\operatorname{Re}\partial_\alpha\grave{A}^a_\gamma}\right)\operatorname{Re}\grave{A}^a_\beta + \frac{d}{d\xi^\gamma}\left(\frac{d\mathcal{L}}{d\operatorname{Im}\partial_\alpha\grave{A}^a_\gamma}\operatorname{Im}\grave{A}^a_\beta\right)$
$-\left(\frac{d}{d\xi^\gamma}\frac{d\mathcal{L}}{d\operatorname{Im}\partial_\alpha\grave{A}^a_\gamma}\right)\operatorname{Im}\grave{A}^a_\beta + \frac{d}{d\xi^\gamma}\left(\frac{d\mathcal{L}}{d\partial_\alpha G^a_\gamma}G^a_\beta\right) - \left(\frac{d}{d\xi^\gamma}\frac{d\mathcal{L}}{d\partial_\alpha G^a_\gamma}\right)G^a_\beta\}$
$-\sum_i\left(\frac{d\mathcal{L}}{dO'^i_\alpha}O'^i_\beta + \sum_\gamma\frac{d}{d\xi^\gamma}\left(\frac{d\mathcal{L}}{d\partial_\gamma O^{i\prime}_\alpha}O'^i_\beta\right) - \left(\frac{d}{d\xi^\gamma}\frac{d\mathcal{L}}{d\partial_\gamma O^{i\prime}_\alpha}\right)O'^i_\beta\right)$
$= -\sum_a\left(\frac{d\mathcal{L}}{dG^a_\alpha} - \frac{d}{d\xi^\gamma}\frac{d\mathcal{L}}{d\partial_\gamma G^a_\alpha}\right)G^a_\beta - \sum_i\left(\left(\frac{d\mathcal{L}}{dO'^i_\alpha} - \frac{d}{d\xi^\gamma}\frac{d\mathcal{L}}{d\partial_\gamma O^{i\prime}_\alpha}\right)O'^i_\beta\right)$
$+\sum_a\left(\frac{d\mathcal{L}}{d\operatorname{Re}\grave{A}^a_\alpha} - \frac{d}{d\xi^\gamma}\frac{d\mathcal{L}}{d\operatorname{Re}\partial_\gamma\grave{A}^a_\alpha}\right)\operatorname{Re}\grave{A}^a_\beta + \left(\frac{d\mathcal{L}}{d\operatorname{Im}\grave{A}^a_\alpha} - \frac{d}{d\xi^\gamma}\frac{d\mathcal{L}}{d\operatorname{Im}\partial_\gamma\grave{A}^a_\alpha}\right)\operatorname{Im}\grave{A}^a_\beta\}$
$+\sum_\gamma\frac{d}{d\xi^\gamma}\left(\sum_a\frac{d\mathcal{L}}{d\operatorname{Re}\partial_\alpha\grave{A}^a_\gamma}\operatorname{Re}\grave{A}^a_\beta + \frac{d\mathcal{L}}{d\operatorname{Im}\partial_\alpha\grave{A}^a_\gamma}\operatorname{Im}\grave{A}^a_\beta + \frac{d\mathcal{L}}{d\partial_\alpha G^a_\gamma}G^a_\beta - \sum_i\frac{d\mathcal{L}}{d\partial_\gamma O^{i\prime}_\alpha}O'^i_\beta\right)$
where we used : $\frac{d\mathcal{L}}{d\partial_\gamma G^a_\alpha} = -\frac{d\mathcal{L}}{d\partial_\alpha G^a_\gamma}$, $\frac{d\mathcal{L}}{d\operatorname{Re}\partial_\gamma\grave{A}^a_\alpha} = -\frac{d\mathcal{L}}{d\operatorname{Re}\partial_\alpha\grave{A}^a_\gamma}$, $\frac{d\mathcal{L}}{d\operatorname{Im}\partial_\gamma\grave{A}^a_\alpha} = -\frac{d\mathcal{L}}{d\operatorname{Im}\partial_\alpha\grave{A}^a_\gamma}$
$Y^\alpha_{H\beta}\det O' = -\sum_a\left(\frac{\delta S}{\delta G^a_\alpha}G^a_\beta + \frac{\delta S}{\delta\operatorname{Re}\grave{A}^a_\alpha}\operatorname{Re}\grave{A}^a_\beta + \frac{\delta\mathcal{L}}{\delta\operatorname{Im}\grave{A}^a_\alpha}\operatorname{Im}\grave{A}^a_\beta\right) - \sum_i\frac{\delta S}{\delta O'^i_\alpha}O'^i_\beta$
$+\sum_\gamma\frac{d}{d\xi^\gamma}\left(\sum_a\frac{d\mathcal{L}}{d\operatorname{Re}\partial_\gamma\grave{A}^a_\alpha}\operatorname{Re}\grave{A}^a_\beta + \frac{d\mathcal{L}}{d\operatorname{Im}\partial_\gamma\grave{A}^a_\alpha}\operatorname{Im}\grave{A}^a_\beta + \frac{d\mathcal{L}}{d\partial_\gamma G^a_\alpha}G^a_\beta + \sum_i\frac{d\mathcal{L}}{d\partial_\gamma O^{i\prime}_\alpha}O'^i_\beta\right)$
Thus on shell :
$Y^\alpha_{H\beta}\det O'$
$= \sum_\gamma\frac{d}{d\xi^\gamma}\left(\sum_a\frac{d\mathcal{L}}{d\operatorname{Re}\partial_\gamma\grave{A}^a_\alpha}\operatorname{Re}\grave{A}^a_\beta + \frac{d\mathcal{L}}{d\operatorname{Im}\partial_\gamma\grave{A}^a_\alpha}\operatorname{Im}\grave{A}^a_\beta + \frac{d\mathcal{L}}{d\partial_\gamma G^a_\alpha}G^a_\beta + \sum_i\frac{d\mathcal{L}}{d\partial_\gamma O^{i\prime}_\alpha}O'^i_\beta\right)$
that hints at some kind of exterior derivative of a form. The trouble comes from $\frac{d\mathcal{L}}{d\partial_\gamma O^{i\prime}_\alpha}$ which is not an antisymmetric 2-vector.

10.2 Superpotential

1) Let be the quantities

$Z^{\alpha\gamma}_{H\beta} = \sum_a \frac{dL}{d\operatorname{Re}\partial_\alpha \grave{A}^a_\gamma} \operatorname{Re}\grave{A}^a_\beta + \frac{dL}{d\operatorname{Im}\partial_\alpha \grave{A}^a_\gamma} \operatorname{Im}\grave{A}^a_\beta + \frac{dL}{d\partial_\alpha G^a_\gamma} G^a_\beta + \sum_i \left(\frac{dL}{d\partial_\alpha O'^i_\gamma} - \frac{dL}{d\partial_\gamma O'^i_\alpha} \right) O'^i_\beta.$

and keep β fixed. They are the components of an antisymmetric 2-vector field $Z_{H\beta} = \sum_{\{\alpha\gamma\}} Z^{\alpha\gamma}_{H\beta} \partial_\alpha \wedge \partial_\gamma$ on M. Indeed in a change of chart we have (see section "covariance") :

$\widehat{Z^{\alpha\gamma}_{H\beta}} = \sum_{\lambda\mu\nu} \sum_a K^\alpha_\lambda K^\gamma_\mu \frac{dL}{d\operatorname{Re}\partial_\lambda \grave{A}^a_\mu} \operatorname{Re} J^\nu_\beta \grave{A}^a_\nu + K^\alpha_\lambda K^\gamma_\mu \frac{dL}{d\operatorname{Im}\partial_\lambda \grave{A}^a_\mu} \operatorname{Im} J^\nu_\beta \grave{A}^a_\nu$

$+ K^\alpha_\lambda K^\gamma_\mu \frac{dL}{d\partial_\lambda G^a_\mu} J^\nu_\beta G^a_\nu + \sum_i \left(K^\alpha_\lambda K^\gamma_\mu \frac{dL}{d\partial_\lambda O'^i_\mu} - K^\gamma_\lambda K^\alpha_\mu \frac{dL}{d\partial_\mu O'^i_\lambda} \right) J^\nu_\beta O'^i_\nu$

$\widehat{Z^{\alpha\gamma}_{H\beta}} = \sum_{\lambda\mu\nu} K^\alpha_\lambda K^\gamma_\mu J^\nu_\beta Z^{\lambda\mu}_{H\nu}$

We have :

$\sum_{\{\alpha\gamma\}} Z^{\alpha\gamma}_{H\beta} \partial_\alpha \wedge \partial_\gamma$

$= \sum_{\{\alpha\gamma\}} \{ \sum_a \left(\frac{dL}{d\operatorname{Re}\partial_\alpha \grave{A}^a_\gamma} \partial_\alpha \wedge \partial_\gamma \right) \operatorname{Re}\grave{A}^a_\beta + \left(\frac{dL}{d\operatorname{Im}\partial_\alpha \grave{A}^a_\gamma} \partial_\alpha \wedge \partial_\gamma \right) \operatorname{Im}\grave{A}^a_\beta$

$+ \left(\frac{dL}{d\partial_\alpha G^a_\gamma} \partial_\alpha \wedge \partial_\gamma \right) G^a_\beta + \sum_i \left(\left(\frac{dL}{d\partial_\alpha O'^i_\gamma} - \frac{dL}{d\partial_\gamma O'^i_\alpha} \right) \partial_\alpha \wedge \partial_\gamma \right) O'^i_\beta \}$

$Z_{H\beta} = \sum_a Z^a_{AR} \operatorname{Re}\grave{A}^a_\beta + Z^a_{AI} \operatorname{Im}\grave{A}^a_\beta + Z^a_G G^a_\beta + \sum_i O'^i_\beta Z^i_O$

with $Z^i_O = \sum_{\{\alpha\gamma\}} \left(\frac{dL}{d\partial_\alpha O'^i_\gamma} - \frac{dL}{d\partial_\gamma O'^i_\alpha} \right) \partial_\alpha \wedge \partial_\gamma$

2) Compute the superpotential $\Pi_{H\beta} = \varpi_4 (Z_{H\beta})$. By the same calculation as above we get :

$\Pi_{H\beta} = \varpi_4 (Z_{H\beta}) = 2 (\det O') \sum_{\lambda<\mu} \sum_{\alpha<\gamma} \epsilon (\lambda, \mu, \alpha, \gamma) Z^{\alpha\gamma}_{H\beta} \left(dx^\lambda \wedge dx^\mu \right)$ is a 2-form on M :

$\Pi_{H\beta} = -2 (\det O') \sum_a \{ Z^{32}_{H\beta} dx^0 \wedge dx^1 + Z^{13}_{H\beta} dx^0 \wedge dx^2 + Z^{21}_{H\beta} dx^0 \wedge dx^3 + Z^{03}_{H\beta} dx^2 \wedge dx^1 + Z^{02}_{H\beta} dx^1 \wedge dx^3 + Z^{01}_{H\beta} dx^3 \wedge dx^2 \}$

$\Pi_{H\beta} = \sum_a \left(\operatorname{Re}\grave{A}^a_\beta \right) \Pi^a_{AR} + \left(\operatorname{Im}\grave{A}^a_\beta \right) \Pi^a_{AI} + G^a_\beta \Pi^a_G + \sum_i O'^i_\beta \Pi^i_O$

with $\Pi^i_O = \varpi_4 (Z^i_O)$

Its exterior derivative is a 3-form over M :

$d\Pi_{H\beta}$

$= -2 \sum_{\alpha=0}^3 (-1)^{\alpha+1} \left(\sum_{\gamma=0}^3 \partial_\gamma \left(Z^{\alpha\gamma}_{H\beta} \det O' \right) \right) dx^0 \wedge ... \wedge \widehat{dx^\alpha} \wedge ... \wedge dx^3$

$= -2 \frac{1}{\det O'} \varpi_4 \left(\sum_{\alpha\gamma} \left(\partial_\gamma \left(Z^{\alpha\gamma}_{H\beta} \det O' \right) \right) \partial_\alpha \right)$

$= 2 \frac{1}{\det O'} \varpi_4 \left(\sum_{\alpha\gamma} \left(\partial_\gamma \left(Z^{\gamma\alpha}_{H\beta} \det O' \right) \right) \partial_\alpha \right)$

$= 2 \sum_{\alpha=0}^3 (-1)^{\alpha+1} \left(\sum_{\gamma=0}^3 \partial_\gamma \left(Z^{\gamma\alpha}_{H\beta} \det O' \right) \right) dx^0 \wedge ... \wedge \widehat{dx^\alpha} \wedge ... \wedge dx^3$

that we can write : $d\Pi_{H\beta} = 2\frac{1}{\det O'}\varpi_4\left(\sum_{\alpha\gamma}\left(\frac{d}{d\xi^\gamma}\left(Z_{H\beta}^{\gamma\alpha}\det O'\right)\right)\partial_\alpha\right)$ keeping in mind that when the derivative involves L_M we must take the composite function with f.

$d\Pi_{H\beta} = \frac{2}{\det O'}\times$

$\varpi_4\left(\sum_{\alpha\gamma}\frac{d}{d\xi^\gamma}\left(\begin{array}{c}\sum_a \frac{d\mathcal{L}}{d\operatorname{Re}\partial_\gamma\grave{A}_\alpha^a}\operatorname{Re}\grave{A}_\beta^a + \frac{d\mathcal{L}}{d\operatorname{Im}\partial_\gamma\grave{A}_\alpha^a}\operatorname{Im}\grave{A}_\beta^a \\ +\frac{d\mathcal{L}}{d\partial_\gamma G_\alpha^a}G_\beta^a + \sum_i\left(\frac{d\mathcal{L}}{d\partial_\gamma O_\alpha^{i\prime}} - \frac{d\mathcal{L}}{d\partial_\alpha O_\gamma^{i\prime}}\right)O_\beta^{\prime i}\end{array}\right)\right)dx^0\wedge..\widehat{dx^\alpha}..dx^3$

10.3 Conservation law

The value of $\varpi_4(Y_{H\beta})$ is the 3-form over M :

$\varpi_4(Y_{H\beta}) = \sum_{\alpha=0}^3(-1)^{\alpha+1}Y_{H\beta}^\alpha\det O' dx^0\wedge...\wedge\widehat{dx^\alpha}\wedge...\wedge dx^3$

On shell we have :

$\sum_{\alpha=0}^3(-1)^{\alpha+1}Y_{H\beta}^\alpha(\det O')\,dx^0\wedge...\widehat{dx^\alpha}..\wedge dx^3 = \varpi_4(Y_{H\beta})$

$= \sum_{\alpha,\gamma=0}^3(-1)^{\alpha+1}\frac{d}{d\xi^\gamma}(\sum_a\frac{d\mathcal{L}}{d\operatorname{Re}\partial_\gamma\grave{A}_\alpha^a}\operatorname{Re}\grave{A}_\beta^a + \frac{d\mathcal{L}}{d\operatorname{Im}\partial_\gamma\grave{A}_\alpha^a}\operatorname{Im}\grave{A}_\beta^a$

$+\frac{d\mathcal{L}}{d\partial_\gamma G_\alpha^a}G_\beta^a + \sum_i\frac{d\mathcal{L}}{d\partial_\gamma O_\alpha^{i\prime}}O_\beta^{\prime i})dx^0\wedge..\widehat{dx^\alpha}..dx^3$

$\varpi_4(Y_{H\beta})$

$= \varpi_4\left(\sum_{\alpha\gamma}\frac{d}{d\xi^\gamma}(\sum_a\frac{d\mathcal{L}}{d\operatorname{Re}\partial_\gamma\grave{A}_\alpha^a}\operatorname{Re}\grave{A}_\beta^a + \frac{d\mathcal{L}}{d\operatorname{Im}\partial_\gamma\grave{A}_\alpha^a}\operatorname{Im}\grave{A}_\beta^a + \frac{d\mathcal{L}}{d\partial_\gamma G_\alpha^a}G_\beta^a + \sum_i\frac{d\mathcal{L}}{d\partial_\gamma O_\alpha^{i\prime}}O_\beta^{\prime i})\partial_\alpha\right)$

$= \varpi_4(\sum_{\alpha\gamma}\frac{d}{d\xi^\gamma}(\sum_a\frac{d\mathcal{L}}{d\operatorname{Re}\partial_\gamma\grave{A}_\alpha^a}\operatorname{Re}\grave{A}_\beta^a + \frac{d\mathcal{L}}{d\operatorname{Im}\partial_\gamma\grave{A}_\alpha^a}\operatorname{Im}\grave{A}_\beta^a + \frac{d\mathcal{L}}{d\partial_\gamma G_\alpha^a}G_\beta^a$

$+\sum_i\left(\frac{d\mathcal{L}}{d\partial_\gamma O_\alpha^{i\prime}} - \frac{d\mathcal{L}}{d\partial_\alpha O_\gamma^{i\prime}}\right)O_\beta^{\prime i})\partial_\alpha) + \varpi_4\left(\sum_{\alpha\gamma}\frac{d}{d\xi^\gamma}\left(\sum_i\frac{d\mathcal{L}}{d\partial_\alpha O_\gamma^{i\prime}}O_\beta^{\prime i}\right)\partial_\alpha\right)$

$= \frac{1}{2}d\Pi_{H\beta} + \varpi_4\left(\sum_{\alpha\gamma}\frac{d}{d\xi^\gamma}\left(\sum_i\frac{d\mathcal{L}}{d\partial_\alpha O_\gamma^{i\prime}}O_\beta^{\prime i}\right)\partial_\alpha\right)$

$$\varpi_4(Y_{H\beta}) = \frac{1}{2}(d\Pi_{H\beta} + \Pi_{O\beta}) \tag{65}$$

with $\Pi_{O\beta} = \sum_{\alpha,\gamma=0}^3(-1)^{\alpha+1}\frac{d}{d\xi^\gamma}\left(\sum_i\frac{d\mathcal{L}}{d\partial_\alpha O_\gamma^{i\prime}}O_\beta^{\prime i}\right)dx^0\wedge...\wedge\widehat{dx^\alpha}\wedge...\wedge dx^3$

$= \frac{1}{\det O'}\varpi_4\left(\sum_{\alpha\gamma i}\left(\frac{d}{d\xi^\gamma}\left(\frac{d\mathcal{L}}{d\partial_\alpha O_\gamma^{i\prime}}O_\beta^{\prime i}\right)\right)\partial_\alpha\right)$

If the equations related to the force fields are met, this latter equation is equivalent to the "frame equation".

$d(\varpi_4(Y_{H\beta})) = \left(\sum_{\alpha,\gamma=0}^3\frac{d^2}{d\xi^\alpha d\xi^\gamma}\left(\sum_i\frac{dL}{d\partial_\alpha O_\gamma^{i\prime}}O_\beta^{\prime i}\det O'\right)\right)\varpi_0$

The flow of the $Y_{H\beta}$ vector is conserved if this quantity is null, which is met if L does not depend on $\partial_\alpha O_\gamma^{i\prime}$ or if $\frac{dL}{d\partial_\alpha O_\gamma^{i\prime}} = -\frac{dL}{d\partial_\gamma O_\alpha^{i\prime}}$. But we will see a stronger result.

10.4 The superconservation law

It is intuitive that there is some relation between all the Noether currents and potentials, looking like an energy conservation law. In addition neither the term L in Y_H or the Π_O quantity in the latest equation are too appealing. We can give a more convenient formula for this equation 65, but that will require some work, that is, in some ways, a reverse engineering of what has been done above.

We will prove that :

$\forall \alpha, \beta : Y^{\alpha}_{H\beta} = 0$

$d\left(\Pi_{H\beta}\right) = -\frac{2}{\det O'} \sum_i \varpi_4 \left(\sum_{\alpha\gamma} \frac{d}{d\xi^{\gamma}} \left(O'^{i}_{\beta} \frac{d\mathcal{L}}{d\partial_{\alpha} O'^{i\prime}_{\gamma}} \right) \partial_{\alpha} \right) = -2\Pi_{O\beta}$

meaning that the pertinent physical quantity is the energy-momentum tensor $\delta^{\alpha}_{\beta} L$. And the second equations gives, in the usual case where the lagrangian does not depend of the derivatives $\partial_{\alpha} O^{i\prime}_{\gamma}$, a general law linking the gravitational and the other force fields, without any involvement of the particles.

1) We will start by expliciting $\delta^{\alpha}_{\beta} L$ on shell, that will be useful later.

$\delta^{\alpha}_{\beta} L$ is given by the frame equation :

$\forall \alpha, \beta : \delta^{\alpha}_{\beta} L = -\sum_i \frac{dL}{dO'^{i}_{\alpha}} O'^{i}_{\beta} + \frac{1}{\det O'} \sum_{i\gamma} O'^{i}_{\beta} \frac{d}{d\xi^{\gamma}} \left(\frac{dL \det O'}{d\partial_{\gamma} O'^{i}_{\alpha}} \right)$

Using the covariance equations 43,44 as above we get :

$\sum_i \frac{dL_M}{dO'^{i}_{\alpha}} O'^{i}_{\beta} = \frac{dL_M}{dV^{\beta}} V^{\alpha} - \sum_{i,j} \left(\frac{dL_M}{d \operatorname{Re} \nabla_{\alpha} \psi^{ij}} \operatorname{Re} \nabla_{\beta} \psi^{ij} + \frac{dL_M}{d \operatorname{Im} \nabla_{\alpha} \psi^{ij}} \operatorname{Im} \nabla_{\beta} \psi^{ij} \right)$

$-\sum_a \frac{\partial L_M}{\partial G^{a}_{\alpha}} G^{a}_{\beta} - \sum_{i\lambda} \frac{dL_M}{d\partial_{\lambda} O'^{i}_{\alpha}} \left(\partial_{\lambda} O'^{i}_{\beta} \right) + \frac{dL_M}{d\partial_{\alpha} O'^{i}_{\lambda}} \left(\partial_{\beta} O'^{i}_{\lambda} \right)$

$\sum_i \frac{dL_F}{dO'^{i}_{\alpha}} O'^{i}_{\beta} = -2 \sum_{a\lambda} \left(\frac{dL_F}{d \operatorname{Re} \mathcal{F}^{a}_{A,\alpha\lambda}} \operatorname{Re} \mathcal{F}^{a}_{A,\beta\lambda} + \frac{dL_F}{d \operatorname{Im} \mathcal{F}^{a}_{A,\alpha\lambda}} \operatorname{Im} \mathcal{F}^{a}_{A,\beta\lambda} + \frac{dL_F}{d\mathcal{F}^{a}_{G,\alpha\lambda}} \mathcal{F}^{a}_{G,\beta\lambda} \right)$

$-\sum_a \frac{\partial L_F}{\partial G^{a}_{\alpha}} G^{a}_{\beta} - \sum_{i\lambda} \frac{dL_F}{d\partial_{\lambda} O'^{i\prime}_{\alpha}} \left(\partial_{\lambda} O'^{i}_{\beta} \right) + \frac{dL_F}{d\partial_{\alpha} O'^{i\prime}_{\lambda}} \left(\partial_{\beta} O'^{i}_{\lambda} \right)$

$\sum_i \frac{dL}{dO'^{i}_{\alpha}} O'^{i}_{\beta} = \frac{dNL_M}{dV^{\beta}} V^{\alpha} - \sum_{i,j} \left(\frac{dNL_M}{d \operatorname{Re} \nabla_{\alpha} \psi^{ij}} \operatorname{Re} \nabla_{\beta} \psi^{ij} + \frac{dNL_M}{d \operatorname{Im} \nabla_{\alpha} \psi^{ij}} \operatorname{Im} \nabla_{\beta} \psi^{ij} \right)$

$-2 \sum_{a\lambda} \left(\frac{dL_F}{d \operatorname{Re} \mathcal{F}^{a}_{A,\alpha\lambda}} \operatorname{Re} \mathcal{F}^{a}_{A,\beta\lambda} + \frac{dL_F}{d \operatorname{Im} \mathcal{F}^{a}_{A,\alpha\lambda}} \operatorname{Im} \mathcal{F}^{a}_{A,\beta\lambda} + \frac{dL_F}{d\mathcal{F}^{a}_{G,\alpha\lambda}} \mathcal{F}^{a}_{G,\beta\lambda} \right)$

$-\sum_a \frac{\partial L}{\partial G^{a}_{\alpha}} G^{a}_{\beta} - \sum_{i\lambda} \frac{dL}{d\partial_{\lambda} O'^{i}_{\alpha}} \partial_{\lambda} O'^{i}_{\beta} + \frac{dL}{d\partial_{\alpha} O'^{i}_{\lambda}} \partial_{\beta} O'^{i}_{\lambda}$

$\delta^{\alpha}_{\beta} L = \sum_{i,j} \left(\frac{dNL_M}{d \operatorname{Re} \nabla_{\alpha} \psi^{ij}} \operatorname{Re} \nabla_{\beta} \psi^{ij} + \frac{dNL_M}{d \operatorname{Im} \nabla_{\alpha} \psi^{ij}} \operatorname{Im} \nabla_{\beta} \psi^{ij} \right)$

$+2 \sum_{a\gamma} \left(\frac{dL_F}{d \operatorname{Re} \mathcal{F}^{a}_{A,\alpha\gamma}} \operatorname{Re} \mathcal{F}^{a}_{A,\beta\gamma} + \frac{dL_F}{d \operatorname{Im} \mathcal{F}^{a}_{A,\alpha\gamma}} \operatorname{Im} \mathcal{F}^{a}_{A,\beta\gamma} + \frac{dL_F}{d\mathcal{F}^{a}_{G,\alpha\gamma}} \mathcal{F}^{a}_{G,\beta\gamma} \right)$

$+ \sum_a \frac{\partial L}{\partial G^{a}_{\alpha}} G^{a}_{\beta} + \sum_{i\gamma} \frac{dL}{d\partial_{\gamma} O'^{i}_{\alpha}} \partial_{\gamma} O'^{i}_{\beta} + \frac{dL}{d\partial_{\alpha} O'^{i}_{\gamma}} \partial_{\beta} O'^{i}_{\gamma} + \frac{1}{\det O'} \sum_{i\gamma} O'^{i}_{\beta} \frac{d}{d\xi^{\gamma}} \left(\frac{dL \det O'}{d\partial_{\gamma} O'^{i}_{\alpha}} \right) - \frac{dNL_M}{dV^{\beta}} V^{\alpha}$

Let us expand the first term.

$\sum_{i,j}\left(\frac{dNL_M}{d\operatorname{Re}\nabla_\alpha\psi^{ij}}\operatorname{Re}\nabla_\beta\psi^{ij}+\frac{dNL_M}{d\operatorname{Im}\nabla_\alpha\psi^{ij}}\operatorname{Im}\nabla_\beta\psi^{ij}\right)$

$=\sum_{i,j}\left(\frac{dNL_M}{d\operatorname{Re}\nabla_\alpha\psi^{ij}}\operatorname{Re}\partial_\beta\psi^{ij}+\frac{dNL_M}{d\operatorname{Im}\nabla_\alpha\psi^{ij}}\operatorname{Im}\partial_\beta\psi^{ij}\right)$

$+\sum_{aij}\frac{dNL_M}{d\operatorname{Re}\nabla_\alpha\psi^{ij}}\operatorname{Re}\left(G^a_\beta\left[\kappa_a\right]\left[\psi\right]+\grave{A}^a_\beta\left[\psi\right]\left[\theta_a\right]^t\right)^{ij}$

$+\frac{dNL_M}{d\operatorname{Im}\nabla_\alpha\psi^{ij}}\operatorname{Im}\left(G^a_\beta\left[\kappa_a\right]\left[\psi\right]+\grave{A}^a_\beta\left[\psi\right]\left[\theta_a\right]^t\right)^{ij}$

$=\sum_{i,j}\left(\frac{dNL_M}{d\operatorname{Re}\nabla_\alpha\psi^{ij}}\operatorname{Re}\partial_\beta\psi^{ij}+\frac{dNL_M}{d\operatorname{Im}\nabla_\alpha\psi^{ij}}\operatorname{Im}\partial_\beta\psi^{ij}\right)$

$+\sum_a G^a_\beta\sum_{ij}\frac{dNL_M}{d\operatorname{Re}\nabla_\alpha\psi^{ij}}\operatorname{Re}\left(\left[\kappa_a\right]\left[\psi\right]\right)^{ij}+\frac{dNL_M}{d\operatorname{Im}\nabla_\alpha\psi^{ij}}\operatorname{Im}\left(\left[\kappa_a\right]\left[\psi\right]\right)^{ij}$

$+\sum_a\left(\operatorname{Re}\grave{A}^a_\beta\right)\left(\frac{dNL_M}{d\operatorname{Re}\nabla_\alpha\psi^{ij}}\operatorname{Re}\left(\left[\psi\right]\left[\theta_a\right]^t\right)^{ij}+\frac{dNL_M}{d\operatorname{Im}\nabla_\alpha\psi^{ij}}\operatorname{Im}\left(\left[\psi\right]\left[\theta_a\right]^t\right)^{ij}\right)$

$+\left(\operatorname{Im}\grave{A}^a_\beta\right)\left(-\frac{dNL_M}{d\operatorname{Re}\nabla_\alpha\psi^{ij}}\operatorname{Im}\left(\left[\psi\right]\left[\theta_a\right]^t\right)^{ij}+\frac{dNL_M}{d\operatorname{Im}\nabla_\alpha\psi^{ij}}\operatorname{Re}\left(\left[\psi\right]\left[\theta_a\right]^t\right)^{ij}\right)$

And with the Noether currents it reads :

$\sum_{i,j}\left(\frac{dNL_M}{d\operatorname{Re}\nabla_\alpha\psi^{ij}}\operatorname{Re}\nabla_\beta\psi^{ij}+\frac{dNL_M}{d\operatorname{Im}\nabla_\alpha\psi^{ij}}\operatorname{Im}\nabla_\beta\psi^{ij}\right)$

$=\sum_{i,j}\left(\frac{dNL_M}{d\operatorname{Re}\nabla_\alpha\psi^{ij}}\operatorname{Re}\partial_\beta\psi^{ij}+\frac{dNL_M}{d\operatorname{Im}\nabla_\alpha\psi^{ij}}\operatorname{Im}\partial_\beta\psi^{ij}\right)$

$+\sum_a G^a_\beta\left(Y^{a\alpha}_G-\frac{\partial L}{\partial G^a_\alpha}-2\sum_{b,\gamma}\frac{dL_F}{d\mathcal{F}^b_{G\alpha\gamma}}\left[\overrightarrow{\kappa}_a,G_\gamma\right]^b\right)$

$+\sum_a\left(\operatorname{Re}\grave{A}^a_\beta\right)\left(Y^{\alpha a}_{AR}-2\sum_{b\gamma}\frac{dL_F}{d\operatorname{Re}\mathcal{F}^b_{A,\alpha\gamma}}\operatorname{Re}\left[\overrightarrow{\theta}_a,\grave{A}_\gamma\right]^b+\frac{dL_F}{d\operatorname{Im}\mathcal{F}^b_{A,\alpha\gamma}}\operatorname{Im}\left[\overrightarrow{\theta}_a,\grave{A}_\gamma\right]^b\right)$

$+\left(\operatorname{Im}\grave{A}^a_\beta\right)\left(Y^{\alpha a}_{AI}-2\sum_{b\gamma}-\frac{dL_F}{d\operatorname{Re}\mathcal{F}^b_{A,\alpha\gamma}}\operatorname{Im}\left[\overrightarrow{\theta}_a,\grave{A}_\gamma\right]^b+\frac{dL_F}{d\operatorname{Im}\mathcal{F}^b_{A,\alpha\gamma}}\operatorname{Re}\left[\overrightarrow{\theta}_a,\grave{A}_\gamma\right]^b\right)$

$=\sum_{i,j}\left(\frac{dNL_M}{d\operatorname{Re}\nabla_\alpha\psi^{ij}}\operatorname{Re}\partial_\beta\psi^{ij}+\frac{dNL_M}{d\operatorname{Im}\nabla_\alpha\psi^{ij}}\operatorname{Im}\partial_\beta\psi^{ij}\right)+\sum_a G^a_\beta\left(Y^{a\alpha}_G-\frac{\partial L}{\partial G^a_\alpha}\right)$

$+Y^{\alpha a}_{AR}\operatorname{Re}\grave{A}^a_\beta+Y^{\alpha a}_{AI}\operatorname{Im}\grave{A}^a_\beta-2\sum_{a,\gamma}\frac{dL_F}{d\mathcal{F}^b_{G\alpha\gamma}}\left[G_\beta,G_\gamma\right]^a$

$-2\sum_{a\gamma}\left(\frac{dL_F}{d\operatorname{Re}\mathcal{F}^a_{A,\alpha\gamma}}\operatorname{Re}\left[\grave{A}_\beta,\grave{A}_\gamma\right]^a+\frac{dL_F}{d\operatorname{Im}\mathcal{F}^a_{A,\alpha\gamma}}\operatorname{Im}\left[\grave{A}_\beta,\grave{A}_\gamma\right]^a\right)$

Thus :

$\delta^\alpha_\beta L=\sum_{i,j}\left(\frac{dNL_M}{d\operatorname{Re}\nabla_\alpha\psi^{ij}}\operatorname{Re}\partial_\beta\psi^{ij}+\frac{dNL_M}{d\operatorname{Im}\nabla_\alpha\psi^{ij}}\operatorname{Im}\partial_\beta\psi^{ij}\right)$

$+\sum_a G^a_\beta\left(Y^{a\alpha}_G-\frac{\partial L}{\partial G^a_\alpha}\right)+Y^{\alpha a}_{AR}\operatorname{Re}\grave{A}^a_\beta+Y^{\alpha a}_{AI}\operatorname{Im}\grave{A}^a_\beta-2\sum_{a,\gamma}\frac{dL_F}{d\mathcal{F}^b_{G\alpha\gamma}}\left[G_\beta,G_\gamma\right]^a$

$-2\sum_{a\gamma}\left(\frac{dL_F}{d\operatorname{Re}\mathcal{F}^a_{A,\alpha\gamma}}\operatorname{Re}\left[\grave{A}_\beta,\grave{A}_\gamma\right]^a+\frac{dL_F}{d\operatorname{Im}\mathcal{F}^a_{A,\alpha\gamma}}\operatorname{Im}\left[\grave{A}_\beta,\grave{A}_\gamma\right]^a\right)$

$+2\sum_{a\gamma}\left(\frac{dL_F}{d\operatorname{Re}\mathcal{F}^a_{A,\alpha\gamma}}\operatorname{Re}\mathcal{F}^a_{A,\beta\gamma}+\frac{dL_F}{d\operatorname{Im}\mathcal{F}^a_{A,\alpha\gamma}}\operatorname{Im}\mathcal{F}^a_{A,\beta\gamma}+\frac{dL_F}{d\mathcal{F}^a_{G,\alpha\gamma}}\mathcal{F}^a_{G,\beta\gamma}\right)$

$+\sum_a\frac{\partial L}{\partial G^a_\alpha}G^a_\beta+\sum_{i\gamma}\frac{dL}{d\partial_\gamma O'^i_\alpha}\partial_\gamma O'^i_\beta+\frac{dL}{d\partial_\alpha O'^i_\gamma}\partial_\beta O'^i_\gamma$

$+\frac{1}{\det O'}\sum_{i\gamma}O'^i_\beta\frac{d}{d\xi^\gamma}\left(\frac{dL\det O'}{d\partial_\gamma O'^i_\alpha}\right)-\frac{dNL_M}{dV^\beta}V^\alpha$

$\delta^\alpha_\beta L = \sum_{i,j}\left(\frac{dNL_M}{d\operatorname{Re}\nabla_\alpha\psi^{ij}}\operatorname{Re}\partial_\beta\psi^{ij} + \frac{dNL_M}{d\operatorname{Im}\nabla_\alpha\psi^{ij}}\operatorname{Im}\partial_\beta\psi^{ij}\right)$
$+\sum_a G^a_\beta Y^{a\alpha}_G + Y^{\alpha a}_{AR}\operatorname{Re}\grave{A}^a_\beta + Y^{\alpha a}_{AI}\operatorname{Im}\grave{A}^a_\beta + 2\sum_{a,\gamma}\frac{dL_F}{d\mathcal{F}^a_{G\alpha\gamma}}\left(\mathcal{F}_{G,\beta\gamma} - [G_\beta, G_\gamma]\right)^a$
$+2\sum_{a\gamma}\left(\frac{dL_F}{d\operatorname{Re}\mathcal{F}^a_{A,\alpha\gamma}}\operatorname{Re}\left(\mathcal{F}_{A,\beta\gamma} - \left[\grave{A}_\beta, \grave{A}_\gamma\right]\right)^a + \frac{dL_F}{d\operatorname{Im}\mathcal{F}^a_{A,\alpha\gamma}}\operatorname{Im}\left(\mathcal{F}_{A,\beta\gamma} - \left[\grave{A}_\beta, \grave{A}_\gamma\right]\right)^a\right)$
$+\sum_{i\gamma}\frac{dL}{d\partial_\gamma O'^i_\alpha}\partial_\gamma O'^i_\beta + \frac{dL}{d\partial_\alpha O'^i_\gamma}\partial_\beta O'^i_\gamma + \frac{1}{\det O'}\sum_{i\gamma}\frac{d}{d\xi^\gamma}\left(\frac{dL\det O'}{d\partial_\gamma O'^i_\alpha}\right) - \frac{dNL_M}{dV^\beta}V^\alpha$

With the same conventions as above for derivation with composite functions :

$\sum_{i\gamma}\frac{dL}{d\partial_\gamma O'^i_\alpha}\partial_\gamma O'^i_\beta + \frac{1}{\det O'}\sum_{i\gamma}O'^i_\beta\frac{d}{d\xi^\gamma}\left(\frac{dL\det O'}{d\partial_\gamma O'^i_\alpha}\right)$
$= \frac{1}{\det O'}\sum_{i\gamma}\frac{d}{d\xi^\gamma}\left(\frac{dL\det O'}{d\partial_\gamma O'^i_\alpha}O'^i_\beta\right) = \frac{1}{\det O'}\sum_{i\gamma}\frac{d}{d\xi^\gamma}\left(\frac{d\mathcal{L}}{d\partial_\gamma O'^i_\alpha}O'^i_\beta\right)$

$\delta^\alpha_\beta L$
$= \sum_{i,j}\left(\frac{dNL_M}{d\operatorname{Re}\nabla_\alpha\psi^{ij}}\operatorname{Re}\partial_\beta\psi^{ij} + \frac{dNL_M}{d\operatorname{Im}\nabla_\alpha\psi^{ij}}\operatorname{Im}\partial_\beta\psi^{ij}\right)$
$+\sum_a G^a_\beta Y^{a\alpha}_G + Y^{\alpha a}_{AR}\operatorname{Re}\grave{A}^a_\beta + Y^{\alpha a}_{AI}\operatorname{Im}\grave{A}^a_\beta$
$+2\sum_{a\gamma}(\frac{dL_F}{d\mathcal{F}^b_{G\alpha\gamma}}\left(\partial_\beta G_\gamma - \partial_\gamma G_\beta\right)^a + \frac{dL_F}{d\operatorname{Re}\mathcal{F}^a_{A,\alpha\gamma}}\operatorname{Re}\left(\partial_\beta\grave{A}_\gamma - \partial_\gamma\grave{A}_\beta\right)^a$
$+\frac{dL_F}{d\operatorname{Im}\mathcal{F}^a_{A,\alpha\gamma}}\operatorname{Im}\left(\partial_\beta\grave{A}_\gamma - \partial_\gamma\grave{A}_\beta\right)^a)$
$+\sum_{i\gamma}\frac{dL}{d\partial_\alpha O'^i_\gamma}\partial_\beta O'^i_\gamma + \frac{1}{\det O'}\sum_{i\gamma}\frac{d}{d\xi^\gamma}\left(\frac{d\mathcal{L}}{d\partial_\gamma O'^i_\alpha}O'^i_\beta\right) - \frac{dNL_M}{dV^\beta}V^\alpha$

2) So on shell $Y^\alpha_{H\beta}$ can be written :

$Y^\alpha_{H\beta} = -\frac{dNL_M}{dV^\beta}V^\alpha + \sum_{ij}\frac{dNL_M}{d\operatorname{Re}\nabla_\alpha\psi^{ij}}\operatorname{Re}\partial_\beta\psi^{ij} + \frac{dNL_M}{d\operatorname{Im}\nabla_\alpha\psi^{ij}}\operatorname{Im}\partial_\beta\psi^{ij}$
$+2\sum_{a,\gamma}\frac{dL_F}{d\mathcal{F}^a_{G\alpha\gamma}}\partial_\beta G^a_\gamma + \frac{dL_F}{d\operatorname{Re}\mathcal{F}^a_{A\alpha\gamma}}\operatorname{Re}\partial_\beta\grave{A}^a_\gamma + \frac{dL_F}{d\operatorname{Im}\mathcal{F}^a_{A\alpha\gamma}}\operatorname{Im}\partial_\beta\grave{A}^a_\gamma + \sum_{i\gamma}\frac{dL}{d\partial_\alpha O'^i_\gamma}\partial_\beta O'^i_\gamma$
$-\sum_{i,j}\left(\frac{dVL_M}{d\operatorname{Re}\nabla_\alpha\psi^{ij}}\operatorname{Re}\partial_\beta\psi^{ij} + \frac{dVL_M}{d\operatorname{Im}\nabla_\alpha\psi^{ij}}\operatorname{Im}\partial_\beta\psi^{ij}\right)$
$-\sum_a G^a_\beta Y^{a\alpha}_G + Y^{\alpha a}_{AR}\operatorname{Re}\grave{A}^a_\beta + Y^{\alpha a}_{AI}\operatorname{Im}\grave{A}^a_\beta$
$-2\sum_{a\gamma}(\frac{dL_F}{d\mathcal{F}^b_{G\alpha\gamma}}\left(\partial_\beta G_\gamma - \partial_\gamma G_\beta\right)^a + \frac{dL_F}{d\operatorname{Re}\mathcal{F}^a_{A,\alpha\gamma}}\operatorname{Re}\left(\partial_\beta\grave{A}_\gamma - \partial_\gamma\grave{A}_\beta\right)^a$
$+\frac{dL_F}{d\operatorname{Im}\mathcal{F}^a_{A,\alpha\gamma}}\operatorname{Im}\left(\partial_\beta\grave{A}_\gamma - \partial_\gamma\grave{A}_\beta\right)^a)$
$-\frac{dL}{d\partial_\alpha O'^i_\gamma}\partial_\beta O'^i_\gamma - \frac{1}{\det O'}\sum_{i\gamma}\frac{d}{d\xi^\gamma}\left(\frac{d\mathcal{L}}{d\partial_\gamma O'^i_\alpha}O'^i_\beta\right) + \frac{dNL_M}{dV^\beta}V^\alpha$
$Y^\alpha_{H\beta} = -\sum_a\left(G^a_\beta Y^{a\alpha}_G + Y^{\alpha a}_{AR}\operatorname{Re}\grave{A}^a_\beta + Y^{\alpha a}_{AI}\operatorname{Im}\grave{A}^a_\beta\right)$
$+2\sum_{a\gamma}\left(\frac{dL_F}{d\mathcal{F}^b_{G\alpha\gamma}}\partial_\gamma G^a_\beta + \frac{dL_F}{d\operatorname{Re}\mathcal{F}^a_{A,\alpha\gamma}}\operatorname{Re}\partial_\gamma\grave{A}^a_\beta + \frac{dL_F}{d\operatorname{Im}\mathcal{F}^a_{A,\alpha\gamma}}\operatorname{Im}\partial_\gamma\grave{A}^a_\beta\right)$
$-\frac{1}{\det O'}\sum_{i\gamma}\frac{d}{d\xi^\gamma}\left(\frac{d\mathcal{L}}{d\partial_\gamma O'^i_\alpha}O'^i_\beta\right)$

This formula will be improved.

3) Taking the value of $\varpi_4(Y_{H\beta})$:

$\varpi_4(Y_{H\beta})$

$= -\sum_a \left(G^a_\beta \varpi_4(Y^a_G) + \left(\operatorname{Re}\grave{A}^a_\beta\right)\varpi_4(Y^\alpha_{AR}) + \left(\operatorname{Im}\grave{A}^a_\beta\right)\varpi_4(Y^\alpha_{AI})\right)$

$+\sum_a \varpi_4\left(\left(\sum_\gamma 2\frac{dL}{d\mathcal{F}^a_{G\alpha\gamma}}\partial_\gamma G^a_\beta\right)\partial_\alpha\right) + \varpi_4\left(\left(\sum_\gamma 2\frac{dL_F}{d\operatorname{Re}\mathcal{F}^a_{A\alpha\gamma}}\operatorname{Re}\left(\partial_\gamma \grave{A}^a_\beta\right)\right)\partial_\alpha\right)$

$+\varpi_4\left(\left(\sum_\gamma 2\frac{dL_F}{d\operatorname{Im}\mathcal{F}^a_{A\alpha\gamma}}\operatorname{Im}\left(\partial_\gamma \grave{A}^a_\beta\right)\right)\partial_\alpha\right) - \frac{1}{\det O'}\sum_i \varpi_4\left(\sum_{\alpha\gamma}\left(\frac{d}{d\xi^\gamma}\left(\frac{d\mathcal{L}}{d\partial_\gamma O'^i_\alpha}O'^i_\beta\right)\right)\partial_\alpha\right)$

With the usual algebraic calculation :

$\varpi_4\left(\left(\sum_\gamma 2\frac{dL}{d\mathcal{F}^a_{G\alpha\gamma}}\partial_\gamma G^a_\beta\right)\partial_\alpha\right) = \varpi_4\left(\left(\sum_\gamma Z^{a\alpha\gamma}_G \partial_\gamma G^a_\beta\right)\partial_\alpha\right)$

$= -\frac{1}{2}\left(\sum_\gamma \partial_\gamma G^a_\beta dx^\gamma\right)\wedge \Pi^a_G = -\frac{1}{2}d\left(G^a_\beta\right)\wedge\Pi^a_G$

$\varpi_4\left(\left(\sum_\gamma 2\frac{dL_F}{d\operatorname{Re}\mathcal{F}^a_{A\alpha\gamma}}\operatorname{Re}\left(\partial_\gamma \grave{A}^a_\beta\right)\right)\partial_\alpha\right) = -\frac{1}{2}d\left(\operatorname{Re}\grave{A}^a_\beta\right)\wedge\Pi^a_{AR}$

$\varpi_4\left(\left(\sum_\gamma 2\frac{dL_F}{d\operatorname{Im}\mathcal{F}^a_{A\alpha\gamma}}\operatorname{Im}\left(\partial_\gamma \grave{A}^a_\beta\right)\right)\partial_\alpha\right) = -\frac{1}{2}d\left(\operatorname{Im}\grave{A}^a_\beta\right)\wedge\Pi^a_{AI}$

$\varpi_4(Y_{H\beta}) = -\sum_a \left(G^a_\beta \varpi_4(Y^a_G) + \left(\operatorname{Re}\grave{A}^a_\beta\right)\varpi_4(Y^\alpha_{AR}) + \left(\operatorname{Im}\grave{A}^a_\beta\right)\varpi_4(Y^\alpha_{AI})\right)$

$-\frac{1}{2}d\left(G^a_\beta\right)\wedge\Pi^a_G - \frac{1}{2}d\left(\operatorname{Re}\grave{A}^a_\beta\right)\wedge\Pi^a_{AR} - \frac{1}{2}d\left(\operatorname{Im}\grave{A}^a_\beta\right)\wedge\Pi^a_{AI}$

$-\frac{1}{\det O'}\sum_i \varpi_4\left(\sum_{\alpha\gamma}\left(\frac{d}{d\xi^\gamma}\left(\frac{d\mathcal{L}}{d\partial_\gamma O'^i_\alpha}O'^i_\beta\right)\right)\partial_\alpha\right)$

And on shell we have from the other conservation equations :

$\varpi_4(Y^a_G) = \frac{1}{2}d\Pi^a_G, \varpi_4(Y^\alpha_{AR}) = \frac{1}{2}d\Pi^a_{AR}, \varpi_4(Y^\alpha_{AI}) = \frac{1}{2}d\Pi^a_{AI}$

$\varpi_4(Y_{H\beta})$

$= -\frac{1}{2}\sum_a G^a_\beta d\Pi^a_G + d\left(G^a_\beta\right)\wedge\Pi^a_G + \left(\operatorname{Re}\grave{A}^a_\beta\right)d\Pi^a_{AR} + d\left(\operatorname{Re}\grave{A}^a_\beta\right)\wedge\Pi^a_{AR}$

$+\left(\operatorname{Im}\grave{A}^a_\beta\right)d\Pi^a_{AI} + d\left(\operatorname{Im}\grave{A}^a_\beta\right)\wedge\Pi^a_{AI} - \frac{1}{\det O'}\sum_i \varpi_4\left(\sum_{\alpha\gamma}\left(\frac{d}{d\xi^\gamma}\left(\frac{d\mathcal{L}}{d\partial_\gamma O'^i_\alpha}O'^i_\beta\right)\right)\partial_\alpha\right)$

$\varpi_4(Y_{H\beta}) = -\frac{1}{2}\sum_a d\left(G^a_\beta\Pi^a_G\right) + d\left(\left(\operatorname{Re}\grave{A}^a_\beta\right)\Pi^a_{AR}\right) + d\left(\left(\operatorname{Im}\grave{A}^a_\beta\right)\Pi^a_{AI}\right)$

$-\frac{1}{\det O'}\sum_i \varpi_4\left(\sum_{\alpha\gamma}\left(\frac{d}{d\xi^\gamma}\left(\frac{d\mathcal{L}}{d\partial_\gamma O'^i_\alpha}O'^i_\beta\right)\right)\partial_\alpha\right)$

4) The superpotential is :

$\Pi_{H\beta} = \sum_a \left(\operatorname{Re}\grave{A}^a_\beta\right)\Pi^a_{AR} + \left(\operatorname{Im}\grave{A}^a_\beta\right)\Pi^a_{AI} + G^a_\beta\Pi^a_G + \sum_i O'^i_\beta \Pi^i_O$

with $\Pi^i_O = \varpi_4\left(Z^i_O\right), Z^i_O = \sum_i \left(\frac{dL}{d\partial_\alpha O'^i_\gamma} - \frac{dL}{d\partial_\gamma O'^i_\alpha}\right)\partial_\alpha \wedge \partial_\gamma$

and the conservation equation 65

$\varpi_4(Y_{H\beta}) = \frac{1}{2}d\Pi_{H\beta} + \frac{1}{\det O'}\varpi_4\left(\sum_{\alpha\gamma i}\left(\frac{d}{d\xi^\gamma}\left(\frac{d\mathcal{L}}{d\partial_\alpha O'^i_\gamma}O'^i_\beta\right)\right)\partial_\alpha\right)$

becomes :

$-\frac{1}{2}\sum_a d\left(G^a_\beta\Pi^a_G\right) + d\left(\left(\operatorname{Re}\grave{A}^a_\beta\right)\Pi^a_{AR}\right) + d\left(\left(\operatorname{Im}\grave{A}^a_\beta\right)\Pi^a_{AI}\right)$

$-\frac{1}{\det O'}\sum_i \varpi_4\left(\sum_{\alpha\gamma}\left(\frac{d}{d\xi^\gamma}\left(\frac{d\mathcal{L}}{d\partial_\gamma O'^i_\alpha}O'^i_\beta\right)\right)\partial_\alpha\right)$
$=\frac{1}{2}\sum_a d\left(G^a_\beta\Pi^a_G\right)+d\left(\left(\operatorname{Re}\grave{A}^a_\beta\right)\Pi^a_{AR}\right)+d\left(\left(\operatorname{Im}\grave{A}^a_\beta\right)\Pi^a_{AI}\right)$
$+\frac{1}{2}\sum_i d\left(O'^i_\beta\Pi^i_O\right)+\frac{1}{\det O'}\varpi_4\left(\sum_{\alpha\gamma i}\left(\frac{d}{d\xi^\gamma}\left(\frac{d\mathcal{L}}{d\partial_\alpha O'^{i\prime}_\gamma}O'^i_\beta\right)\right)\partial_\alpha\right)$
That is :
$\sum_a d\left(G^a_\beta\Pi^a_G\right)+d\left(\left(\operatorname{Re}\grave{A}^a_\beta\right)\Pi^a_{AR}\right)+d\left(\left(\operatorname{Im}\grave{A}^a_\beta\right)\Pi^a_{AI}\right)$
$=-\frac{1}{2}\sum_i d\left(O'^i_\beta\Pi^i_O\right)-\frac{1}{\det O'}\sum_i \varpi_4\left(\sum_{\alpha\gamma}\frac{d}{d\xi^\gamma}\left(O'^i_\beta\left(\frac{d\mathcal{L}}{d\partial_\gamma O'^i_\alpha}+\frac{d\mathcal{L}}{d\partial_\alpha O'^{i\prime}_\gamma}\right)\right)\partial_\alpha\right)$
Or :
$d\left(\Pi_{H\beta}\right)=\sum_i\left(\frac{1}{2}d\left(O'^i_\beta\Pi^i_O\right)-\frac{1}{\det O'}\varpi_4\left(\sum_{\alpha\gamma}\frac{d}{d\xi^\gamma}\left(O'^i_\beta\left(\frac{d\mathcal{L}}{d\partial_\gamma O'^i_\alpha}+\frac{d\mathcal{L}}{d\partial_\alpha O'^{i\prime}_\gamma}\right)\right)\partial_\alpha\right)\right)$
But :
$d\left(O'^i_\beta\Pi^i_O\right)=\left(dO'^i_\beta\right)\wedge\Pi^i_O+O'^i_\beta d\Pi^i_O$
$\varpi_4\left(\sum_{\alpha\gamma}\left(Z^{i\alpha\gamma}_O\frac{d}{d\xi^\gamma}O'^i_\beta\right)\partial_\alpha\right)$
$=-\frac{1}{2}\left(\sum_{\alpha\gamma}\left(\frac{d}{d\xi^\gamma}O'^i_\beta\right)dx^\gamma\right)\wedge\Pi^a_O=-\frac{1}{2}dO'^i_\beta\wedge\Pi^a_O$
$d\Pi^i_O=-2\frac{1}{\det O'}\sum_a\varpi_4\left(\sum_{\alpha\gamma}\frac{d}{d\xi^\gamma}\left(Z^{i\alpha\gamma}_O\det O'\right)\partial_\alpha\right)$
$d\left(\Pi_{H\beta}\right)=\sum_i\{-\frac{1}{\det O'}O'^i_\beta\varpi_4\left(\sum_{\alpha\gamma}\frac{d}{d\xi^\gamma}\left(Z^{i\alpha\gamma}_O\det O'\right)\partial_\alpha\right)$
$-\varpi_4\left(\sum_{\alpha\gamma}\left(Z^{i\alpha\gamma}_O\frac{d}{d\xi^\gamma}O'^i_\beta\right)\partial_\alpha\right)$
$-\frac{1}{\det O'}\varpi_4\left(\sum_{\alpha\gamma}\frac{d}{d\xi^\gamma}\left(O'^i_\beta\left(\frac{d\mathcal{L}}{d\partial_\gamma O'^i_\alpha}+\frac{d\mathcal{L}}{d\partial_\alpha O'^{i\prime}_\gamma}\right)\right)\partial_\alpha\right)\}$
$=-\frac{1}{\det O'}\sum_i\varpi_4\left(\sum_{\alpha\gamma}\begin{pmatrix}O'^i_\beta\frac{d}{d\xi^\gamma}\left(Z^{i\alpha\gamma}_O\det O'\right)+Z^{i\alpha\gamma}_O\det O'\frac{d}{d\xi^\gamma}O'^i_\beta\\+\frac{d}{d\xi^\gamma}\left(O'^i_\beta\left(\frac{d\mathcal{L}}{d\partial_\gamma O'^i_\alpha}+\frac{d\mathcal{L}}{d\partial_\alpha O'^{i\prime}_\gamma}\right)\right)\end{pmatrix}\partial_\alpha\right)$
$=-\frac{1}{\det O'}\sum_i\varpi_4\sum_{\alpha\gamma}\frac{d}{d\xi^\gamma}\left(O'^i_\beta\left(\frac{d\mathcal{L}}{d\partial_\alpha O'^{i\prime}_\gamma}-\frac{d\mathcal{L}}{d\partial_\gamma O'^i_\alpha}+\frac{d\mathcal{L}}{d\partial_\gamma O'^i_\alpha}+\frac{d\mathcal{L}}{d\partial_\alpha O'^{i\prime}_\gamma}\right)\right)\partial_\alpha$
$=-\frac{2}{\det O'}\sum_i\varpi_4\left(\sum_{\alpha\gamma}\frac{d}{d\xi^\gamma}\left(O'^i_\beta\frac{d\mathcal{L}}{d\partial_\alpha O'^{i\prime}_\gamma}\right)\partial_\alpha\right)=-2\Pi_{O\beta}$

$$\mathbf{d}\left(\Pi_{H\beta}\right)=-\frac{2}{\det O'}\sum_i\varpi_4\left(\sum_{\alpha\gamma}\frac{d}{d\xi^\gamma}\left(O'^i_\beta\frac{d\mathcal{L}}{d\partial_\alpha O'^{i\prime}_\gamma}\right)\partial_\alpha\right)=-\mathbf{2\Pi}_{O\beta}\quad(66)$$

5) Therefore :
$\varpi_4\left(Y_{H\beta}\right)=\frac{1}{2}d\Pi_{H\beta}+\Pi_{O\beta}=0$
As :
$\varpi_4\left(Y_{H\beta}\right)=\sum_{\alpha=0}^3\left(-1\right)^{\alpha+1}Y^\alpha_{H\beta}\det O'dx^0\wedge...\wedge\widehat{dx^\alpha}\wedge...\wedge dx^3$

we have :

$$\forall \alpha, \beta : \mathbf{Y}^{\alpha}_{H\beta} = \mathbf{0} \tag{67}$$

6) And :
$\frac{1}{\det O'} \sum_{i\gamma} \frac{d}{d\xi^{\gamma}} \left(\frac{d\mathcal{L}}{d\partial_{\gamma} O'^{i}_{\alpha}} O'^{i}_{\beta} \right) = -\sum_{a} \left(G^{a}_{\beta} Y^{a\alpha}_{G} + Y^{\alpha a}_{AR} \operatorname{Re} \grave{A}^{a}_{\beta} + Y^{\alpha a}_{AI} \operatorname{Im} \grave{A}^{a}_{\beta} \right)$
$+2 \sum_{a\gamma} \left(\frac{dL_F}{d\mathcal{F}^{b}_{G\alpha\gamma}} \partial_{\gamma} G^{a}_{\beta} + \frac{dL_F}{d \operatorname{Re} \mathcal{F}^{a}_{A,\alpha\gamma}} \operatorname{Re} \partial_{\gamma} \grave{A}^{a}_{\beta} + \frac{dL_F}{d \operatorname{Im} \mathcal{F}^{a}_{A,\alpha\gamma}} \operatorname{Im} \partial_{\gamma} \grave{A}^{a}_{\beta} \right)$
So :
$\delta^{\alpha}_{\beta} L = \sum_{i,j} \left(\frac{dVL_M}{d \operatorname{Re} \nabla_{\alpha} \psi^{ij}} \operatorname{Re} \partial_{\beta} \psi^{ij} + \frac{dVL_M}{d \operatorname{Im} \nabla_{\alpha} \psi^{ij}} \operatorname{Im} \partial_{\beta} \psi^{ij} \right)$
$+ \sum_{a} G^{a}_{\beta} Y^{a\alpha}_{G} + Y^{\alpha a}_{AR} \operatorname{Re} \grave{A}^{a}_{\beta} + Y^{\alpha a}_{AI} \operatorname{Im} \grave{A}^{a}_{\beta}$
$+2 \sum_{a\gamma} (\frac{dL_F}{d\mathcal{F}^{b}_{G\alpha\gamma}} \left(\partial_{\beta} G_{\gamma} - \partial_{\gamma} G_{\beta} \right)^{a} + \frac{dL_F}{d \operatorname{Re} \mathcal{F}^{a}_{A,\alpha\gamma}} \operatorname{Re} \left(\partial_{\beta} \grave{A}_{\gamma} - \partial_{\gamma} \grave{A}_{\beta} \right)^{a}$
$+ \frac{dL_F}{d \operatorname{Im} \mathcal{F}^{a}_{A,\alpha\gamma}} \operatorname{Im} \left(\partial_{\beta} \grave{A}_{\gamma} - \partial_{\gamma} \grave{A}_{\beta} \right)^{a})$
$+ \sum_{i\gamma} \frac{dL}{d\partial_{\alpha} O'^{i}_{\gamma}} \partial_{\beta} O'^{i}_{\gamma} - \sum_{a} \left(G^{a}_{\beta} Y^{a\alpha}_{G} + Y^{\alpha a}_{AR} \operatorname{Re} \grave{A}^{a}_{\beta} + Y^{\alpha a}_{AI} \operatorname{Im} \grave{A}^{a}_{\beta} \right)$
$+2 \sum_{a\gamma} \left(\frac{dL_F}{d\mathcal{F}^{a}_{G\alpha\gamma}} \partial_{\gamma} G^{a}_{\beta} + \frac{dL_F}{d \operatorname{Re} \mathcal{F}^{a}_{A,\alpha\gamma}} \operatorname{Re} \partial_{\gamma} \grave{A}^{a}_{\beta} + \frac{dL_F}{d \operatorname{Im} \mathcal{F}^{a}_{A,\alpha\gamma}} \operatorname{Im} \partial_{\gamma} \grave{A}^{a}_{\beta} \right) - \frac{dNL_M}{dV^{\beta}} V^{\alpha}$

We get a formula, valid on shell, for the quantity $\delta^{\alpha}_{\beta} L$ that we name the energy-momentum tensor.

$$\begin{aligned} \delta^{\alpha}_{\beta} L = & -\frac{dNL_M}{dV^{\beta}} V^{\alpha} + \sum_{i\gamma} \frac{dL}{d\partial_{\alpha} O'^{i}_{\gamma}} \partial_{\beta} O'^{i}_{\gamma} + \sum_{i,j} \left(\frac{dNL_M}{d \operatorname{Re} \nabla_{\alpha} \psi^{ij}} \operatorname{Re} \partial_{\beta} \psi^{ij} + \frac{dNL_M}{d \operatorname{Im} \nabla_{\alpha} \psi^{ij}} \operatorname{Im} \partial_{\beta} \psi^{ij} \right) \\ & +2 \sum_{a\gamma} \left(\frac{dL_F}{d\mathcal{F}^{a}_{G\alpha\gamma}} \partial_{\beta} G^{a}_{\gamma} + \frac{dL_F}{d \operatorname{Re} \mathcal{F}^{a}_{A,\alpha\gamma}} \operatorname{Re} \partial_{\beta} \grave{A}^{a}_{\gamma} + \frac{dL_F}{d \operatorname{Im} \mathcal{F}^{a}_{A,\alpha\gamma}} \operatorname{Im} \partial_{\beta} \grave{A}^{a}_{\gamma} \right) \end{aligned} \tag{68}$$

10.5 Energy

1) It is useful to give some thoughts about the physical meaning of these results.

If in the equation 68 we put $\alpha = \beta$:
$L = -\frac{dNL_M}{dV^{\alpha}} V^{\alpha} + \sum_{i,j} \left(\frac{dNL_M}{d \operatorname{Re} \nabla_{\alpha} \psi^{ij}} \operatorname{Re} \partial_{\alpha} \psi^{ij} + \frac{dNL_M}{d \operatorname{Im} \nabla_{\alpha} \psi^{ij}} \operatorname{Im} \partial_{\alpha} \psi^{ij} \right)$
$+2 \sum_{a\gamma} \left(\frac{dL_F}{d\mathcal{F}^{a}_{G\alpha\gamma}} \partial_{\alpha} G^{a}_{\gamma} + \frac{dL_F}{d \operatorname{Re} \mathcal{F}^{a}_{A,\alpha\gamma}} \operatorname{Re} \partial_{\alpha} \grave{A}^{a}_{\gamma} + \frac{dL_F}{d \operatorname{Im} \mathcal{F}^{a}_{A,\alpha\gamma}} \operatorname{Im} \partial_{\alpha} \grave{A}^{a}_{\gamma} \right) + \sum_{i\gamma} \frac{dL}{d\partial_{\alpha} O'^{i}_{\gamma}} \partial_{\alpha} O'^{i}_{\gamma}$
As :
$\frac{dL_M}{d \operatorname{Re} \nabla_{\alpha} \psi^{ij}} = \frac{dL}{d \operatorname{Re} \partial_{\alpha} \psi^{ij}}, \frac{dL_M}{d \operatorname{Im} \nabla_{\alpha} \psi^{ij}} = \frac{dL}{d \operatorname{Im} \partial_{\alpha} \psi^{ij}}, 2 \frac{dL_F}{d\mathcal{F}^{b}_{G\alpha\gamma}} = \frac{dL}{d\partial_{\alpha} G^{b}_{\gamma}}, ...$

this equation reads on shell : $\forall \alpha : L = -\frac{dL}{dV^\alpha} V^\alpha + \sum_i \frac{dL}{dz^i_\alpha} \partial_\alpha z^i$

The first term is a kind of "kinetic energy", the next six correspond to the potential energy of the fields, but there is still the last one : $\sum_{i\gamma} \frac{dL}{d\partial_\alpha O'^i_\gamma} \partial_\beta O'^i_\gamma$ which features the distorsion of the space time. We know that energy in General Relativity is a difficult concept. In some way we would expect that the gravitational field encompasses all the effects on the geometry of the universe, and this is why usually one discards the derivatives $\partial_\gamma O'^i_\alpha$, but as we see a more open vision is perhaps necessary. Notice that this issue is not related to the choice of **G** or g as key variable. In the traditional variational version of General Relativity the fundamental term is given by the scalar curvature which is, as seen before, nothing but the curvature form of **G**.

2) The quantity $\int_{\Omega(t)} \varpi_4 (L\partial_\beta)$ is the energy flow through the borders of $\Omega(t)$ seen by an observer on the line ∂_β.The flow is given by $\varpi_4 (L\partial_\beta)$:

$\varpi_4 (L\partial_\beta) = \varpi_4 \left(\left(-\frac{dNL_M}{dV^\beta} V^\alpha + \sum_{i,j} \frac{dNL_M}{d \operatorname{Re} \nabla_\alpha \psi^{ij}} \operatorname{Re} \partial_\beta \psi^{ij} + \frac{dNL_M}{d \operatorname{Im} \nabla_\alpha \psi^{ij}} \operatorname{Im} \partial_\beta \psi^{ij} \right) \partial_\alpha \right)$
$+ \sum_a \varpi_4 \left(2 \sum_{\alpha\gamma} \frac{dL_F}{d\mathcal{F}^a_{G,\alpha\gamma}} \partial_\beta G^a_\gamma \partial_\alpha \right) + \varpi_4 \left(2 \sum_{\alpha\gamma} \frac{dL_F}{d \operatorname{Re} \mathcal{F}^a_{A,\alpha\gamma}} \operatorname{Re} \partial_\beta \grave{A}^a_\gamma \partial_\alpha \right)$
$+\varpi_4 \left(2 \sum_{\alpha\gamma} \frac{dL_F}{d \operatorname{Im} \mathcal{F}^a_{A,\alpha\gamma}} \operatorname{Im} \partial_\beta \grave{A}^a_\gamma \partial_\alpha \right) + \sum_i \varpi_4 \left(\sum_{\alpha\gamma} \frac{dL}{d\partial_\alpha O'^i_\gamma} \partial_\beta O'^i_\gamma \partial_\alpha \right)$
$\varpi_4 (L\partial_\beta) = \varpi_4 \left(\left(-\frac{dNL_M}{dV^\beta} V^\alpha + \sum_{i,j} \frac{dNL_M}{d \operatorname{Re} \nabla_\alpha \psi^{ij}} \operatorname{Re} \partial_\beta \psi^{ij} + \frac{dNL_M}{d \operatorname{Im} \nabla_\alpha \psi^{ij}} \operatorname{Im} \partial_\beta \psi^{ij} \right) \partial_\alpha \right)$
$+ \sum_i \varpi_4 \left(\sum_{\alpha\gamma} \frac{dL}{d\partial_\alpha O'^i_\gamma} \partial_\beta O'^i_\gamma \partial_\alpha \right) - \frac{1}{2} \sum_a \left(\sum_\gamma \partial_\beta G^a_\gamma dx^\gamma \right) \wedge \Pi^a_G + \left(\sum_\gamma \operatorname{Re} \partial_\beta \grave{A}^a_\gamma dx^\gamma \right) \wedge$
$\Pi^a_{AR} + \left(\sum_\gamma \operatorname{Im} \partial_\beta \grave{A}^a_\gamma dx^\gamma \right) \wedge \Pi^a_{AR}$

Part IV
THE MODEL

So far we have shown the constraints imposed on a lagrangian and established the lagrange equations for a not too specific model. In order to improve our grasp of the problem, it is useful to go a step further, and to test our concepts on some lagrangian, keeping it simple enough to enable calculations. As we have seen, in the L_M part of the lagrangian the quantities involved are chiefly the velocity, the state tensors and their covariant derivatives, and in the L_F part they are the curvature forms $\mathcal{F}$. The identities to be met hint at some kind of homogeneous function, so it is legitimate to look for quadratic functions, and thus for scalar products. The scalar products must be defined for the state tensors, on the vector space $F \otimes W$, and for the connections curvatures forms $\mathcal{F}$. They must be invariant in a gauge transformation. If we want to compute a scalar product involving the derivatives of the state tensors, we must find a way to define a differential operator acting on the fiber bundle E_M, it will be the Dirac operator. Eventually we can improve somewhat the definition of the F vector space by introducing chirality.

11 SCALAR PRODUCTS

11.1 Scalar products for the state tensors

1) The first step is to define an hermitian scalar product $\langle\rangle$ on the vector space F, invariant under a gauge transformation by Spin(3,1). An hermitian scalar product on F is represented in the basis e_i by an hermitian matrix A, such that :

$\forall u, v \in F : \langle u, v\rangle = [u]^* [A] [v] ; [A] = [A]^*$

Spin(3,1) acts on F through : $\rho \circ \Upsilon : Spin(3,1) \to L(F;F)$ so A must be such that :

$\forall s \in Spin(3,1) : [\rho (\Upsilon (s))]^* [A] [\rho (\Upsilon (s))] = [A]$

2) The γ matrices are defined up to conjugation by a constant matrix, and any A hermitian matrix meeting the conditions will be defined up to conjugation by an unitary matrix. We choose $A = \gamma_0$. Let us prove that it fits the constraints.

It is an hermitian matrix, as all the other γ_i as we have assumed so far.

$s \in Spin(3,1)$ is the product of an even number of vectors of norm 1 in Cl(3,1) :

$s = v_1 \cdot ..... \cdot v_{2r},$

$v_k = v_1 \varepsilon_1 + v_2 \varepsilon_2 + v_3 \varepsilon_3 + v_0 \varepsilon_0, v_\alpha \in R$

$v_1^2 + v_2^2 + v_3^2 - v_0^2 = 1$

$\Upsilon(v_k) = v_1 \varepsilon_1 + v_2 \varepsilon_2 + v_3 \varepsilon_3 + i v_0 \varepsilon_0$

So $\rho(\Upsilon(s)) = \rho(\Upsilon(v_1)) ... \rho(\Upsilon(v_{2r}))$ and $[\rho(\Upsilon(s))]^* = \rho(\Upsilon(v_{2r}))^* ... \rho(\Upsilon(v_1))^*$

$\rho(\Upsilon(v_k)) = v_1 \gamma_1 + v_2 \gamma_2 + v_3 \gamma_3 + i v_0 \gamma_0$

$\rho(\Upsilon(v_k))^* = v_1 \gamma_1 + v_2 \gamma_2 + v_3 \gamma_3 - i v_0 \gamma_0$ because all the components are real and the γ_k are hermitian

$\rho(\Upsilon(v_k))^* \gamma_0 \rho(\Upsilon(v_k))$

$= (v_1 \gamma_1 + v_2 \gamma_2 + v_3 \gamma_3 - i v_0 \gamma_0) \gamma_0 (v_1 \gamma_1 + v_2 \gamma_2 + v_3 \gamma_3 + i v_0 \gamma_0)$

$= (v_1 \gamma_1 \gamma_0 + v_2 \gamma_2 \gamma_0 + v_3 \gamma_3 \gamma_0 - i v_0 \gamma_0 \gamma_0)(v_1 \gamma_1 + v_2 \gamma_2 + v_3 \gamma_3 + i v_0 \gamma_0)$

$= (-v_1^2 - v_2^2 - v_3^2 + v_0^2) \gamma_0 + v_2 v_1 \gamma_0 \gamma_1 \gamma_2 - v_1 v_2 \gamma_0 \gamma_1 \gamma_2 + v_3 v_1 \gamma_0 \gamma_1 \gamma_3$

$- v_1 v_3 \gamma_0 \gamma_1 \gamma_3 + v_3 v_2 \gamma_0 \gamma_2 \gamma_3 - v_2 v_3 \gamma_0 \gamma_2 \gamma_3$

$+ i v_1 v_0 \gamma_1 - i v_0 v_1 \gamma_1 + i v_2 v_0 \gamma_2 - i v_0 v_2 \gamma_2 + i v_3 v_0 \gamma_3 - i v_0 v_3 \gamma_3$

$= -\gamma_0$

Thus $[\rho(\Upsilon(s))]^* \gamma_0 [\rho(\Upsilon(s))] = (-1)^{2r} \gamma_0 = \gamma_0 \blacksquare$

We will take as scalar product in F :

$u, v \in F : \langle u, v \rangle_F = [u]^* \gamma_0 [v] = \sum_{ij} \gamma_{0ij} \overline{u}^i v^j$

It is not degenerate, but not necessarily definite positive.

Notice that the scalar product is invariant by Spin(3,1), but not Spin(4,C). The basis (e_i) is not necessarily orthonormal : $\langle e_i, e_k \rangle = [\gamma_0]_{ik}$ and the representation $(F, \rho \circ \Upsilon)$ of Spin(3,1) is not necessarily unitary.

From $[\rho(\Upsilon(s))]^* \gamma_0 [\rho(\Upsilon(s))] = \gamma_0$ one deduces by differentiating with respect to s=1 :

$\forall \kappa \in o(3,1) : \frac{d}{ds}\left([\rho(\Upsilon(s))]^* \gamma_0 [\rho(\Upsilon(s))]\right)|_{s=1} = 0$

$\left[\rho(\Upsilon(s))' \Upsilon'(s)\kappa\right]^* \gamma_0 [\rho(\Upsilon(s))] + [\rho(\Upsilon(s))]^* \gamma_0 [\rho'(\Upsilon(s)) \Upsilon'(s) \kappa] = 0$

$\left[\rho(1)' \Upsilon'(1)\kappa\right]^* \gamma_0 + \gamma_4 [\rho'(1) \Upsilon'(1) \kappa] = 0$

$\Upsilon'(1) = Id$

$\left[\rho(1)' \kappa\right]^* \gamma_0 + \gamma_0 [\rho'(1) \kappa] = 0$

Thus with : $\rho'(1) \overrightarrow{\kappa}_a = [\kappa_a] : [\kappa_a]^* \gamma_0 + \gamma_0 [\kappa_a] = 0$

$$[\kappa_a]^* = -[\gamma_0][\kappa_a][\gamma_0] \tag{69}$$

3) We assume that there is an hermitian scalar product on the vector space W, invariant by χ :

$\forall u \in U, \sigma, \sigma' \in W : \langle \sigma, \sigma' \rangle = \langle \chi(u)\sigma, \chi(u)\sigma' \rangle$

and that the basis (f_j) is orthonormal $\langle \sigma, \sigma' \rangle = \sum_i \overline{\sigma}^i \sigma'^i$

Remark : on a complex vector space the signature of an hermitian form can be set up at +.

4) From there we define an hermitian scalar product on $F \otimes W$:

$\left\langle \sigma_1^i \phi_1^j e_i \otimes f_j, \sigma_2^i \phi_2^j e_i \otimes f_j \right\rangle = \left(\sum_{ij} \gamma_{0ij} \overline{\sigma_1}^i \sigma_2^j \right) \left(\sum_k \overline{\phi}_1^k \phi_2^k \right) = \sum_{ijk} \gamma_{0ij} \overline{\sigma_1}^i \overline{\phi}_1^k \sigma_2^j \phi_2^k$

So we define :

$\left\langle \psi_1^{ij} e_i \otimes f_j, \psi_2^{ij} e_i \otimes f_j \right\rangle = \sum_{ijk} \gamma_{0ij} \overline{\psi}_1^{ik} \psi_2^{jk} = [\psi_1^*]_k^i [\gamma_0]_j^i [\psi_2]_k^j = Tr\left([\psi_1]^* \gamma_0 [\psi_2] \right)$

with the 4xm matrices : $[\psi]$

$Tr\overline{\left([\psi_1]^* \gamma_0 [\psi_2] \right)} = Tr\left(\overline{[\psi_1]}^* \overline{\gamma_0}\, \overline{[\psi_2]} \right) = Tr\left([\psi_1]^t \gamma_4^t [\psi_2]^{*t} \right) = Tr\left([\psi_2]^* \gamma_0 [\psi_1] \right)$

And the scalar product is extended on the vector bunle E_M :

$$\psi_1, \psi_2 \in \mathbf{\Lambda}_0(E_v) : \langle \psi_1, \psi_2 \rangle = \mathbf{Tr}\left([\psi_1]^* \gamma_0 [\psi_2] \right) = \sum_{ijk} \gamma_{0ij} \overline{\psi}_1^{ik} \psi_2^{jk} \qquad (70)$$

It should be noticed that this scalar product, as any hermitian scalar product, is invariant by the transformation : $\psi \to z\psi$ where z is c-number valued function on M, with $|z| = 1$.

11.2 Scalar products for the curvature forms

1) The connection and curvature forms are valued in the Lie algebras, and the action of the gauge groups is the adjoint operator Ad, so we need a scalar product on the Lie algebras invariant by the adjoint operator. On any Lie algebra there is a bilinear symmetric form B (the Killing form) invariant by Ad, but non necessarily positive definite. If the Lie algebra is semi-simple (as o(3,1)) it is non degenerate. In the standard representation of o(3,1) : $B(X,Y) = 2Tr([X][Y])$ and its signature is (+ + + - - -) with the basis $(\overrightarrow{\kappa}_a)$.

We will not be so specific and just assume that there is an hermitian scalar product (), invariant by the adjoint action, on the Lie algebras, not necessarily positive definite, for which the bases $(\overrightarrow{\kappa}_a)$, $\left(\overrightarrow{\theta}_a\right)$ are orthogonal :

- o(3,1) : as we use only the real form it is a symmetric real scalar product, and we assume that $(\overrightarrow{\kappa}_a, \overrightarrow{\kappa}_b) = \eta_{ab} = \pm 1$

- T_1U^c :the scalar product is assumed to be hermitian and the basis $\left(\overrightarrow{\theta}_a\right)$ orthonormal : $\left(\overrightarrow{\theta}_a, \overrightarrow{\theta}_b\right) = \delta_{ab} \Rightarrow \left(\overrightarrow{\theta}, \overrightarrow{\theta}'\right) = \sum_a \overline{\theta}^a \theta'^a$

$\forall u \in U, \overrightarrow{\theta}, \overrightarrow{\theta}' \in T_1U^c : \left(Ad_u \overrightarrow{\theta}, Ad_u \overrightarrow{\theta}'\right) = \left(\overrightarrow{\theta}, \overrightarrow{\theta}'\right)$

that implies : $u = \exp \tau \overrightarrow{\theta} : \frac{d}{d\tau} \left(Ad_{\exp \tau \overrightarrow{\theta}} \overrightarrow{\theta}_1, Ad_{\exp \tau \overrightarrow{\theta}} \overrightarrow{\theta}_2\right) |_{\tau=0} = 0 = \left(\left[\overrightarrow{\theta}, \overrightarrow{\theta}_1\right], \overrightarrow{\theta}_2\right) + \left(\overrightarrow{\theta}_1, \left[\overrightarrow{\theta}, \overrightarrow{\theta}_2\right]\right)$

$\Rightarrow \forall \overrightarrow{\theta}, \overrightarrow{\theta}_1, \overrightarrow{\theta}_2 \in T_1U^c : \left(\left[\overrightarrow{\theta}, \overrightarrow{\theta}_1\right], \overrightarrow{\theta}_2\right) = -\left(\overrightarrow{\theta}_1, \left[\overrightarrow{\theta}, \overrightarrow{\theta}_2\right]\right)$

The space vector T_1U^c endowed with this scalar product is a Hilbert space.

2) The potential and the forms are forms on the manifold M. M is endowed with the lorentzian metric g. One defines the scalar product of two r-forms on M valued in $\mathbb{C}$ by :

$G_r : \Lambda_r(M;\mathbb{C}) \times \Lambda_r(M;\mathbb{C}) \to \mathbb{C} ::$

$G_r(\lambda, \mu) = \sum_{\{\alpha_1...\alpha_r\},\{\beta_1...\beta_r\}} \lambda_{\{\alpha_1..\alpha_r\}} \mu_{\{\beta_1...\beta_r\}} \left[\det g^{-1}(x)\right]^{\{\alpha_1...\alpha_r\},\{\beta_1...\beta_r\}}$

$G_r(\lambda, \mu)$

$= \sum_{\{\alpha_1...\alpha_r\}} \lambda_{\{\alpha_1...\alpha_r\}} \sum_{\beta_1...\beta_r} g^{\alpha_1\beta_1}...g^{\alpha_r\beta_r} \mu_{\beta_1...\beta_r} = \sum_{\{\alpha_1...\alpha_r\}} \lambda_{\{\alpha_1...\alpha_r\}} \mu^{\{\alpha_1\alpha_2...\alpha_r\}}$

where the indexes are uppered and lowered with g and $\{\alpha\} = \{\alpha_1...\alpha_r\}$ is an ordered set of r indexes.

The scalar product is symmetric, non degenerated and invariant under a change of chart. It defines an isomorphism between the algebras of r and 4-r forms, given by the Hodge dual.

The Hodge dual λ^* of a r-form $\lambda \in \Lambda_r(M;\mathbb{C})$ is the 4-r form $\lambda^* \in \Lambda_{4-r}(M;\mathbb{C})$ such that :

$\forall \mu \in \Lambda_r(M;\mathbb{R}) : \mu \wedge * \lambda = G_r(\mu, \lambda)\, \varpi_4$ where : $\varpi_4 = (\det O')\, dx^0 \wedge .. \wedge dx^3$ (Taylor [26] 5.8)

$* \left(\sum_{\{\alpha_1...\alpha_r\}} \lambda_{\{\alpha_1...\alpha_r\}} dx^{\alpha_1} .. \wedge dx^{a_r}\right)$

$= \sum_{\{\alpha_1...\alpha_{n-r}\},\{\beta_1...\beta_r\}} \epsilon(\beta_1..\beta_r, \alpha_1, ...\alpha_{n-r})\, u^{\beta_1...\beta_r} (\det O')\, dx^{\alpha_1} .. \wedge dx^{a_{4-r}}$

We have : $** \lambda = (-1)^{r-1} \lambda$

$* \left(\sum_\alpha \lambda_\alpha dx^\alpha\right) = \sum_{\alpha=0}^4 (-1)^{\alpha+1} g^{\alpha\beta} \lambda_\beta (\det O')\, dx^0 \wedge ..\widehat{dx^\alpha} \wedge ...dx^3$

For a 2-form the formula is simple when one uses a convenient ordering of the components.

$\lambda \in \Lambda_2 (M; \mathbb{C})$:
$\lambda = \{\lambda_{01} dx^0 \wedge dx^1 + \lambda_{02} dx^0 \wedge dx^2 + \lambda_{03} dx^0 \wedge dx^3$
$+\lambda_{32} dx^3 \wedge dx^2 + \lambda_{13} dx^1 \wedge dx^3 + \lambda_{21} dx^2 \wedge dx^1\}$

$$(71)$$

$*\lambda = -\{\lambda^{32} dx^0 \wedge dx^1 + \lambda^{13} dx^0 \wedge dx^2 + \lambda^{21} dx^0 \wedge dx^3$
$\quad +\lambda^{01} dx^3 \wedge dx^2 + \lambda^{02} dx^1 \wedge dx^3 + \lambda^{03} dx^2 \wedge dx^1\} \det O'$

Notice that this convenient ordering is deduced from the table 1.
In the orthonormal basis these formulas become :
$\lambda = \sum_{\{i_1..i_r\}} \lambda_{i_1...i_r} \partial^{i_1} \wedge .. \wedge \partial^{i_r}, \mu = \sum_{\{i_1..i_r\}} \mu_{i_1...i_r} \partial^{i_1} \wedge .. \wedge \partial^{i_r}$
$G_r(\lambda, \mu) = \sum_{\{i_1..i_p\}} \eta^{i_1 i_1} ... \eta^{i_p i_p} \lambda_{i_1..i_p} \mu_{i_1..i_p}$ where $\eta^{i_k i_k} = \pm 1$
$*(\sum_{\{i_1..i_p\}} \lambda_{i_1..i_p} \partial^{i_1} \wedge .. \wedge \partial^{i_p})$
$= \sum_{\{i_1..i_p\}, \{j_1..j_{4-p}\}} \epsilon (\{i_1..i_p\}, \{j_1..j_{4-p}\}) \eta_{i_1 i_1} ... \eta_{i_p i_p} \lambda_{\{i_1..i_p\}} \partial^{j_1} \wedge ... \wedge \partial^{j_{4-p}}$
$* \left(\sum_{\{i\}} \lambda_i \partial^i\right) = \sum_{j=0}^3 (-1)^{j+1} \eta_{jj} \lambda_j \partial^0 \wedge .\widehat{\partial^j} .. \wedge \partial^3$
$* \left(\sum_{j=1}^4 \lambda_{0..\widehat{j}...n} \partial^1 \wedge .\widehat{\partial^j} .. \wedge \partial^3\right) = \sum_{j=0}^3 (-1)^{j+1} \eta_{jj} \lambda_{0..\widehat{j}...3} \partial^j$
$* \sum_{\{ij\}} \lambda_{ij} \partial^i \wedge \partial^j$
$= -\lambda_{32} \partial^0 \wedge \partial^1 - \lambda_{13} \partial^0 \wedge \partial^2 - \lambda_{21} \partial^0 \wedge \partial^3 + \lambda_{02} \partial^1 \wedge \partial^3 + \lambda_{01} \partial^3 \wedge \partial^2 + \lambda_{03} \partial^2 \wedge \partial^1$
$* \sum_a \lambda_a \partial^{p_a} \wedge \partial^{q_a} = \sum_{a=1}^3 -\lambda_a \partial^{p_{a+3}} \wedge \partial^{q_{a+3}} + \lambda_{a+3} \partial^{p_a} \wedge \partial^{q_a}$

$$(72)$$

$* (\lambda_{01} \partial^0 \wedge \partial^1 + \lambda_{02} \partial^0 \wedge \partial^2 + \lambda_{03} \partial^0 \wedge \partial^3 + \lambda_{32} \partial^3 \wedge \partial^2 + \lambda_{13} \partial^1 \wedge \partial^3 + \lambda_{21} \partial^2 \wedge \partial^1)$
$= \lambda_{01} \partial^3 \wedge \partial^2 + \lambda_{02} \partial^1 \wedge \partial^3 + \lambda_{03} \partial^2 \wedge \partial^1 - \lambda_{32} \partial^0 \wedge \partial^1 - \lambda_{13} \partial^0 \wedge \partial^2 - \lambda_{21} \partial^0 \wedge \partial^3$

3) Let H_M any vector bundle over M modelled on a vector space H endowed with an hermitian product $()$. One defines the hermitian product of 2 r-forms on M valued in H_M by :

$\lambda, \mu \in \Lambda_r (M; H_M) : \lambda = \sum_a \sum_{\{\alpha\}} \lambda^a_{\{\alpha_1..\alpha_r\}} dx^{\alpha_1} \wedge .. \wedge dx^{\alpha_r} \otimes \overrightarrow{\varkappa}_a (m)$ where $\overrightarrow{\varkappa}_a (m)$ is a local basis of H
$\langle \lambda, \mu \rangle$
$= \sum_{a,b} G_r \left(\overline{\lambda}^a, \mu^b\right) (\overrightarrow{\varkappa}_a, \overrightarrow{\varkappa}_b)$

$= \sum_{a,b} \sum_{\{\alpha_1 \ldots \alpha_r\}} \left(\overline{\lambda}^a_{\{\alpha_1 \ldots \alpha_r\}} \mu^{b,\alpha_1 \alpha_2 \ldots \alpha_r} \right) \left(\overrightarrow{\varkappa}_a, \overrightarrow{\varkappa}_b \right)$
$= \sum_{\{\alpha\},\{\beta\}} \left(\lambda^a_{\{\alpha\}} \overrightarrow{\varkappa}_a, \mu^b_{\{\beta\}} \overrightarrow{\varkappa}_b \right)_H \left[\det g^{-1}(x) \right]^{\{\alpha\},\{\beta\}}$
$= \sum_{a,b} \sum_{\{\alpha_1 \ldots \alpha_r\}} \left(\overline{\lambda}^a_{\{\alpha_1 \ldots \alpha_r\}} \mu^{b,\alpha_1 \alpha_2 \ldots \alpha_r} \right) \left(\overrightarrow{\varkappa}_a, \overrightarrow{\varkappa}_b \right)$
$= \sum_{a,b} \sum_{\{\alpha_1 \ldots \alpha_r\}} \left(\overline{\lambda}^{a,\{\alpha_1 \ldots \alpha_r\}} \mu^b_{\{\alpha_1 \ldots \alpha_r\}} \right) \left(\overrightarrow{\varkappa}_a, \overrightarrow{\varkappa}_b \right)$

It is hermitian, invariant by a chart transformation in M and does not depend of the local basis. If H_M is an associated bundle it is invariant by a gauge transformation.

For the curvature form in an orthonormal basis of $T_1 U^c$:

$$\langle \mathcal{F}_A, \mathcal{F}_A \rangle = \sum_a \sum_{\{\alpha\beta\}} \overline{\mathcal{F}}^a_{A\alpha\beta} \mathcal{F}^{a,\alpha\beta}_A \tag{73}$$

$\langle \mathcal{F}_A, \mathcal{F}_A \rangle$
$= \sum_a \sum_{\{\alpha\beta\}} \overline{\mathcal{F}}^a_{A\alpha\beta} \sum_{\lambda\mu} g^{\alpha\lambda} g^{\beta\mu} \mathcal{F}^a_{A\lambda\mu}$
$= \frac{1}{2} \sum_a \sum_{\alpha\beta\lambda\mu} g^{\alpha\lambda} g^{\beta\mu} \overline{\mathcal{F}}^a_{A\alpha\beta} \mathcal{F}^a_{A\lambda\mu}$
$= \frac{1}{2} \sum_a \sum_{\alpha\beta} \overline{\mathcal{F}}^a_{A\alpha\beta} \mathcal{F}^{a\alpha\beta}_A$

The Hodge dual is defined as : $* \left(\sum_a \lambda^a \otimes \overrightarrow{\varkappa}_a \right) = \sum_a * \overline{\lambda}^a \otimes \overrightarrow{\varkappa}_a$ where $* \overline{\lambda}^a$ is the Hodge dual (in the previous meaning) of the C valued form $\overline{\lambda}^a$

$\lambda, \mu \in \Lambda_r (M; H_M) : \langle \lambda, \mu \rangle = \sum_{a,b} G_r \left(\overline{\lambda}^a, \mu^b \right) \left(\overrightarrow{\varkappa}_a, \overrightarrow{\varkappa}_b \right)$

$\mu^b \wedge * \overline{\lambda}^a = G_r \left(\mu^b, \overline{\lambda}^a \right) \varpi_4 = G_r \left(\overline{\lambda}^a, \mu^b \right) = \mu^b \wedge * \overline{\lambda}^a$

$\langle \lambda, \mu \rangle \varpi_4 = \sum_{a,b} G_r \left(\overline{\lambda}^a, \mu^b \right) \left(\overrightarrow{\varkappa}_a, \overrightarrow{\varkappa}_b \right) \varpi_4 = \sum_{a,b} \left(\mu^b \wedge * \overline{\lambda}^a \right) \left(\overrightarrow{\varkappa}_a, \overrightarrow{\varkappa}_b \right) \varpi_4$

With an orthonormal basis : $\langle \lambda, \mu \rangle \varpi_4 = \sum_a \left(\mu^a \wedge * \overline{\lambda}^a \right) \varpi_4$

For the curvature form :

$$\begin{aligned} * \left(\mathcal{F}_A \right) = - \left(\det O' \right) \textstyle\sum_a \{ & \overline{\mathcal{F}}^{a,01} dx^3 \wedge dx^2 + \overline{\mathcal{F}}^{a,02} dx^1 \wedge dx^3 + \overline{\mathcal{F}}^{a,03} dx^2 \wedge dx^1 \\ & + \overline{\mathcal{F}}^{a,32} dx^0 \wedge dx^1 + \overline{\mathcal{F}}^{a,13} dx^0 \wedge dx^2 + \overline{\mathcal{F}}^{a,21} dx^0 \wedge dx^3 \otimes \overrightarrow{\theta}_a \} \end{aligned} \tag{74}$$

12 THE DIRAC OPERATOR

1) The fields change the state of particles and their velocity. The interaction with the velocity is logically modelled by $\nabla_V \psi = \sum_\alpha V^\alpha \nabla_\alpha \psi$. But we

can assume that there is also an action on the state tensor itself, meaning independant of the dynamic (represented by V). So it is sensible to look after some derivative operator : $\Lambda_0 E_M \to \Lambda_0 E_M$. That is the Dirac operator, which, in matrix form, is : $[D\psi] = \sum_\alpha O_r^\alpha \sum_{r=0}^3 [\gamma^r][\nabla_\alpha \psi]$ where $[\gamma^r]$ are matrices defined from $[\gamma_r]$. The construction is the following.

2) Spin(3,1)xU acts on $Cl(3,1) \times (F \otimes W)$ as : $(s,u) \times (w,\psi) = (\mathbf{Ad}_s w, \vartheta(s,u)\psi)$ so one can define the vector bundle $Q_M \times_{Spin(3,1)\times U} (Cl(3,1) \times (F \otimes W))$ associated to Q_M by the equivalence :

$(q,(w,\psi)) \simeq (q(s^{-1},u^{-1}), (\mathbf{Ad}_s w, \vartheta(s,u)\psi))$

This action is linear with Cl(3,1) and $F \otimes W$ so from there one can get a vector bundle $D_M = Q_M \times_{Spin(3,1)\times U} (Cl(3,1) \otimes F \otimes W))$ associated to Q_M modelled on $Cl(3,1) \otimes F \otimes W$ (the tensor product is associative). The isomorphism :

$(q(m), (\varepsilon_i \otimes e_j \otimes f_k)) \to \varphi_C(m,\varepsilon_i) \otimes e_j(m) \otimes f_k(m) = \partial_i(m) \otimes e_j(m) \otimes f_k(m)$

is preserved by the action of Spin(3,1)xU:

$(q(m)(s^{-1},u^{-1}), ((\mathbf{Ad}_s \varepsilon_i) \otimes \rho \circ \Upsilon(s)(e_j) \otimes \chi(u)(f_k)))$

$\to [\jmath \circ \mu(s)]_i^l \partial_l \otimes \rho(\Upsilon(s))(e_j(m)) \otimes \chi(u)(f_k(m))$

So one can pick $\partial_i(m) \otimes e_j(m) \otimes f_k(m)$ as a basis of this vector bundle D_M.

3) One defines the projection : $P : D_M \to E_M$ by :

$T^{ijk}\partial_i(m) \otimes e_j(m) \otimes f_k(m) \to T^{ijk}(\rho \circ \Upsilon(\varepsilon_i)(e_j))(m) \otimes f_k(m)$

extended by linearity. It is consistent because :

$T^{ijk}\partial_i(m) \otimes e_j(m) \otimes f_k(m)$

$\simeq \left(q(m)(s^{-1},u^{-1}), T^{ijk}(Ad_s\varepsilon_i \otimes \rho \circ \Upsilon(s)(e_j) \otimes \chi(u)(f_k))\right)$

$P\left(\left(q(m)(s^{-1},u^{-1}), T^{ijk}((Ad_s\varepsilon_i) \otimes \rho \circ \Upsilon(s)(e_j) \otimes \chi(u)(f_k))\right)\right)$

$= \left(q(m)(s^{-1},g^{-1}), T^{ijk}(\rho \circ \Upsilon(Ad_s\varepsilon_i)\rho \circ \Upsilon(s)(e_j) \otimes \chi(u)(f_k))\right)$

but

$\rho \circ \Upsilon(Ad_s\varepsilon_i)\rho \circ \Upsilon(s) = \rho \circ \Upsilon(Ad_s\varepsilon_i.s)$

$= \rho \circ \Upsilon(s.\varepsilon_i.s^{-1}.s) = \rho \circ \Upsilon(s.\varepsilon_i) = \rho \circ \Upsilon(s)\rho \circ \Upsilon(\varepsilon_i)$

so :

$\left(q(m)(s^{-1},u^{-1}), T^{ijk}(\rho \circ \Upsilon(s)\rho \circ \Upsilon(\varepsilon_i)(e_j) \otimes \chi(u)(f_k))\right)$

$\simeq \left(q(m), T^{ijk}\rho \circ \Upsilon(\varepsilon_i)e_j \otimes f_k\right) = T^{ijk}(\rho \circ \Upsilon(\varepsilon_i)e_j)(m) \otimes f_k(m)$

4) $\nabla\psi$ is a 1-form on TM*. With the metric we can go from TM* to TM : $dx^{\alpha} \to g^{\alpha\beta}\partial_{\beta}$

$(\nabla_{\alpha}\psi^{ij})\, dx^{\alpha} \otimes e_i\,(m) \otimes f_i(m) \to (\nabla_{\alpha}\psi^{ij})\, g^{\alpha\beta}\partial_{\beta} \otimes e_i\,(m) \otimes f_j\,(m)$

and from TM to D_M :

$\partial_i = O\,(m)_i^{\alpha}\,\partial_{\alpha} \Leftrightarrow \partial_{\alpha} = O\,(m)_{\alpha}^{\prime i}\,\partial_i$

$(\nabla_{\alpha}\psi^{ij})\, g^{\alpha\beta}\partial_{\beta} \otimes e_i\,(m) \otimes f_j\,(m)$

$= (\nabla_{\alpha}\psi^{ij})\, g^{\alpha\beta}O'\,(m)_{\beta}^{l}\,\partial_l \otimes e_i\,(m) \otimes f_j\,(m)$

$= (\nabla_r\psi^{ij})\,\partial_r\,(m) \otimes e_i\,(m) \otimes f_j\,(m)$

The resulting quantity is a section on D_M, that can be projected onto E_M.

So, overall, the Dirac operator is the map $D : \Lambda_0\,(E_M) \to \Lambda_0\,(E_M)$

$D\psi = \sum_{\alpha ij} (\nabla_{\alpha}\psi^{ij}) \sum_{\beta l} g^{\alpha\beta}O'\,(m)_{\beta}^{l}\,(\rho \circ \Upsilon\,(\varepsilon_l)\,(e_i))\,(m) \otimes f_j\,(m)$

5) This quantity can be expressed in a better way. ∂_j is orthonormal so :

$\eta_{ij} = g_{\alpha\beta}\,[O]_i^{\alpha}\,[O]_j^{\beta} \Rightarrow g^{\alpha\beta}\,[O']_{\beta}^{l} = \eta^{lr}\,[O]_r^{\alpha}$

For : l=1,2,3: $\rho \circ \Upsilon\,(\varepsilon_l) = \gamma_l$; and $\rho \circ \Upsilon\,(\varepsilon_0) = i\gamma_0$

$\sum_{\beta l} g^{\alpha\beta}O'\,(m)_{\beta}^{l}\,(\rho \circ \Upsilon\,(\varepsilon_l)\,(e_i))\,(m)$

$= \sum_{rl} \eta^{lr}O_r^{\alpha}\,(\rho \circ \Upsilon\,(\varepsilon_l))_i^p\, e_p\,(m) = \left(\eta^{0r}i\,[\gamma_0]_i^p + \sum_{l=1}^{3} [\gamma_l]_i^p\,\eta^{lr}\right) e_p\,(m)$

We define the matrices : γ^r (index up) such that : $[\gamma^r]_i^p = \eta^{0r}i\,[\gamma_0]_i^p + \sum_{l=1}^{3} [\gamma_l]_i^p\,\eta^{lr}$. We have :

$$\gamma^0 = -\mathbf{i}\gamma_0; \mathbf{r} = \mathbf{1}, \mathbf{2}, \mathbf{3} : \gamma^r = \gamma_r \tag{75}$$

$D\psi = \sum_{\alpha} (\nabla_{\alpha}\psi^{ij}) \sum_r [O]_r^{\alpha}\,[\gamma^r]_i^p\, e_p\,(m) \otimes f_j\,(m)$

We will denote :

$$[\gamma^{\alpha}] = \sum_{r=0}^{3} [O]_r^{\alpha}\,[\gamma^r] \tag{76}$$

$$\sum_{\alpha=0}^{3} \left(\nabla_{\alpha}\psi^{ij}\right) \mathbf{O}_r^{\alpha} = \nabla_{\partial_r}\psi^{ij} = \nabla_r\psi^{ij} \tag{77}$$

With these notations :

$D\psi$
$= \sum_{i,p=1}^{4} \sum_{j=1}^{m} \sum_{r=0}^{3} \left(\nabla_r \psi^{ij}\right) [\gamma^r]_i^p \, e_p(m) \otimes f_j(m)$
$= \sum_{i,p=1}^{4} \sum_{j=1}^{m} \sum_{\alpha=0}^{3} \left(\nabla_\alpha \psi^{ij}\right) [\gamma^\alpha]_i^p \, e_p(m) \otimes f_j(m)$
Or in matrix notation :

$$[D\psi] = \sum_{r=0}^{3} [\gamma^r] [\nabla_r \psi] = \sum_{\alpha=0}^{3} [\gamma^\alpha] [\nabla_\alpha \psi] \tag{78}$$

We see on the relations above that the Dirac operator can be defined through an orthonormal basis only, without any explicit reference to a metric.

6) The Dirac operator is well defined and linear.
In a gauge transformation :
$\left(\nabla_{\partial_r} \psi^{ij}\right) e_i(m) \otimes f_j(m) \simeq \left(\widetilde{\nabla_{\widetilde{\partial}_r} \psi^{ij}}\right) \widetilde{e}_i(m) \otimes \widetilde{f}_j(m)$
$\sum_r \left(\nabla_r \psi^{ij}\right) [\gamma^r]_i^p \, e_p(m) \otimes f_j(m) \simeq \sum_r \left(\widetilde{\nabla_{\widetilde{\partial}_r} \psi^{ij}}\right) [\gamma^r]_i^p \, \widetilde{e}_p(m) \otimes \widetilde{f}_j(m)$
and it is invariant in a change of chart :
$\partial_\alpha \to \widetilde{\partial}_\alpha = J_\alpha^\beta \partial_\beta$
$dx^\alpha \to \widetilde{dx^\alpha} = K_\beta^\alpha dx^\beta$ with $K_\lambda^\alpha J_\beta^\lambda = \delta_\beta^\alpha$
$\nabla_\alpha \psi^{ij} \to \widetilde{\nabla_\alpha \psi^{ij}} = J_\alpha^\lambda \nabla_\lambda \psi^{ij}$
$[O]_r^\alpha \to K_\mu^\alpha [O]_r^\mu$
$\nabla_r \psi^{ij} \to \sum_{\alpha=0}^{3} \left(\widetilde{\nabla_\alpha \psi^{ij}}\right) \widetilde{O}_r^\alpha = \sum_{\alpha=0}^{3} J_\alpha^\lambda \left(\nabla_\lambda \psi^{ij}\right) K_\mu^\alpha [O]_r^\mu$
$= \sum_{\alpha=0}^{3} \left(\nabla_\lambda \psi^{ij}\right) [O]_r^\lambda = \nabla_r \psi^{ij}$
So $D\psi$ is invariant.■

13 CHIRALITY

We have left open the choice of the vector space F in the representation of Cl(4,C). It can be made more precise, without no lost of generality, using a feature of Clifford algebras. It is striking that this feature meets one important characteristic of the physical world, that it distinguishes between the "left" and the "right", that is chirality, and thus is fundamental in particle physics.

13.1 The splitting of Clifford algebras

1) The ordered product $\varepsilon_0 \cdot \varepsilon_1 \cdot .. \cdot \varepsilon_n$ of the vectors of a direct orthonormal basis in a Clifford algebra does not depend of the choice of this basis. In Cl(4,C) $\varepsilon_5 = \varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3$ is such that $(\varepsilon_5)^2 = 1$.

Let be :

$Cl^+(4,C) = \{w \in Cl(4,C) : \varepsilon_5 \cdot w = w\}$,

$Cl^-(4,C) = \{w \in Cl(4,C) : \varepsilon_5 \cdot w = -w\}$

Remind that $Cl(4,C) = Cl_0(4,C) \oplus Cl_1(4,C)$ where $Cl_0(4,C)$ is the subalgebra of the elements sum of an even product of vectors.

Let us prove that $Cl_0(4,C) = Cl_0^+(4,C) \oplus Cl_0^-(4,C)$ where $Cl_0^+(4,C), Cl_0^-(4,C)$ are two subalgebras which are isomorphic and "orthogonal" in that : $\forall w \in Cl_0^+(4,C), w' \in Cl_0^-(4,C) : w \cdot w' = 0$

$Cl_0^+(4,C) = Cl_0(4,C) \cap Cl^+(4,C)$ and $Cl_0^-(4,C) = Cl_0(4,C) \cap Cl^-(4,C)$ are subspace of $Cl_0(4,C)$

$Cl_0(4,C), Cl^+(4,C)$ are subalgebras, so is $Cl_0^+(4,C)$

if $w, w' \in Cl_0^-(4,C), \varepsilon_5 \cdot w \cdot w' = -w \cdot w' \Leftrightarrow w \cdot w' \in Cl_0^-(4,C) : Cl_0^-(4,C)$ is a subalgebra

the only element common to the two subalgebras is 0, thus $Cl_0(4,C) = Cl_0^+(4,C) \oplus Cl_0^-(4,C)$

ε_5 commute with any element of $Cl_0(4,C)$, and anticommute with all elements of $Cl_1(4,C)$ so

If $w \in Cl_0^+(4,C), w' \in Cl_0^-(4,C) : \varepsilon_5 \cdot w = w, \varepsilon_5 \cdot w' = -w'$

$\varepsilon_5 \cdot w \cdot \varepsilon_5 \cdot w' = w \cdot \varepsilon_5 \cdot \varepsilon_5 \cdot w' = w \cdot w' = -\varepsilon_5 \cdot w \cdot w' = -w \cdot w' \Rightarrow w \cdot w' = 0$

■

Similarly : $Cl_1(4,C) = Cl_1^+(4,C) \oplus Cl_1^-(4,C)$ (but they are not subalgebras)

So any element w of Cl(4,C) can be written : $w = w^+ + w^-$ with $w^+ \in Cl^+(4,C), w^- \in Cl^-(4,C)$

2) Let be : $p_\epsilon = \frac{1}{2}(1 + \epsilon\varepsilon_5)$ with $\epsilon = \pm 1$. p_+, p_- are the creation and annihiliation operators

There are the identities :

$p_\epsilon^2 = p_\epsilon, p_+ \cdot p_- = p_- \cdot p_+ = 0, p_+ + p_- = 1$

For any $v \in C^4 : p_\epsilon v = v p_{-\epsilon}$

For any $w = w^+ + w^- \in Cl(4,C) : p_\epsilon \cdot w = \frac{1}{2}((1+\epsilon)w^+ + (1-\epsilon)w^-)$

$\Rightarrow$

$p_+ \cdot w = w^+, p_- \cdot w = w^-; p_+ \cdot w^+ = w^+,$

$p_- \cdot w^+ = 0; p_- \cdot w^- = w^-, p_- \cdot w^+ = 0$
If $w \in Cl_0(4,C) : w \cdot p_\epsilon = \frac{1}{2}((1+\epsilon)w^+ + (1-\epsilon)w^-)$
$\Rightarrow w^+.p_+ = w^+, w^-.p_+ = 0, w^+.p_- = 0, w^-.p_- = w^-$
If $w \in Cl_1(4,C) : w \cdot p_\epsilon = w.\frac{1}{2}(1+\epsilon\varepsilon_5) = \frac{1}{2}(w - \epsilon\varepsilon_5.w)$
$= \frac{1}{2}(w^+ + w^- - \epsilon w^+ + \epsilon w^-) = \frac{1}{2}((1-\epsilon)w^+ + (1+\epsilon)w^-)$
$\Rightarrow w^+.p_+ = 0, w^-.p_+ = w^-, w^+.p_- = w^+, w^-.p_- = 0$

3) We have similarly in the (F,ρ) representation of Cl(4,C) :
$\varepsilon_5 \to \gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$
$p_\epsilon \to \gamma_\epsilon = \frac{1}{2}(I + \epsilon\gamma_5)$
We have the identities :
$\gamma_5^2 = I; \gamma_5 = \gamma_5^*$
$\gamma_+ + \gamma_- = I; \gamma_\epsilon\gamma_\epsilon = \gamma_\epsilon; \gamma_\epsilon\gamma_{-\epsilon} = 0$
k=0..3: $\gamma_5\gamma_k = -\gamma_k\gamma_5; \gamma_\epsilon\gamma_k = \gamma_k\gamma_{-\epsilon};$
$\gamma_+[\kappa_a] = k\gamma_+\gamma_{p_a}\gamma_{q_a} = \frac{1}{2}k\gamma_{p_a}\gamma_{q_a} + \frac{1}{2}k\gamma_5\gamma_{p_a}\gamma_{q_a}$
$= \frac{1}{2}k\gamma_{p_a}\gamma_{q_a} + \frac{1}{2}k\gamma_{p_a}\gamma_{q_a}\gamma_5 = [\kappa_a]\gamma_+$
Let be the vector subspaces :$F^+ = \gamma_+F, F^- = \gamma_-F$
$\gamma_+F^- = \gamma_-F^+ = \{0\} \Rightarrow F^+ \cap F^- = \{0\}$
so : $F = F^+ \oplus F^-$ and $\forall\phi \in F : \phi = \phi^+ + \phi^- = \gamma_+\phi + \gamma_-\phi$

These two subspaces are isomorphic. Indeed for any non null vector v of Cl(4,C):
$p_\epsilon v = vp_{-\epsilon} \Rightarrow \gamma_\epsilon\rho(v) = \rho(v)\gamma_{-\epsilon} \Rightarrow \gamma_\epsilon\rho(v)\phi = \rho(v)\gamma_{-\epsilon}\phi$
$\Rightarrow \gamma_{-\epsilon}\phi = \rho(v)^{-1}\gamma_\epsilon\rho(v)\phi$

For any homogeneous element w of order k in Cl(4,C) :
$w = v_1.....v_k$:
$\rho(w)\gamma_+ = \rho(v_1.....v_k.p_+) = \rho(v_1.....v_{k-1}.p_-.v_k) = \rho(p_\epsilon.v_1.....v_{k-1}v_k) = \gamma_\epsilon\rho(w)$ with $\epsilon = (-1)^k$

and similarly for γ_-. Thus if $w \in Cl_0(4,C)$ we have $\rho(w)\gamma_+ = \gamma_+\rho(w)$ and $\rho(w)\gamma_- = \gamma_-\rho(w)$: F^+ and F^- are globally invariant for the $Cl_0(4,C)$ action. Conversely $Cl_1(4,C)$ exchanges F^+ and F^-.

The Spin(3,1) group image by Υ is $Spin_c(3,1) \subset Cl_0(4,C)$. So the splitting F^+, F^- is stable under the Spin(3,1) action. (F,ρ) is an irreducible representation of the algebra Cl(4,C) but a reducible representation of Spin(3,1) : $(F,\rho) = (F^+,\rho) \oplus (F^-,\rho)$.

4) This splitting is extended in $F \otimes W$. Tensor product being associative we have :

$F\otimes W=(F^{+}\oplus F^{-})\otimes W=(F^{+}\otimes W)\oplus(F^{-}\otimes W)$

and any tensor ψ can be splitted in the sum of two tensors, each one comprised of 2 complex components over F:

$\psi=\psi_{+}+\psi_{-}$ with $\gamma_{+}\psi_{+}=\psi_{+},\gamma_{-}\psi_{-}=\psi_{-},\gamma_{\epsilon}\psi_{-\epsilon}=0$

$\psi=\sum_{ij}\psi^{ij}e_{i}\otimes f_{j}$

$\phi_{j}=\sum_{i}\psi^{ij}e_{i}=\gamma_{+}\phi_{j}+\gamma_{-}\phi_{j}$

$\psi=\sum_{j}(\gamma_{+}\phi_{j}+\gamma_{-}\phi_{j})\otimes f_{j}=\gamma_{+}\left(\sum_{j}\phi_{j}\otimes f_{j}\right)+\gamma_{-}\left(\sum_{j}\phi_{j}\otimes f_{j}\right)=\gamma_{+}\psi+\gamma_{-}\psi;$

$\psi^{ij}=[\gamma_{+}\phi_{j}]^{i}+[\gamma_{-}\phi_{j}]^{i}=[\gamma_{+}]_{k}^{i}\phi_{j}^{k}+[\gamma_{-}]_{k}^{i}\phi_{j}^{k}=[\gamma_{+}]_{k}^{i}\psi^{kj}+[\gamma_{-}]_{k}^{i}\psi^{kj}$

The splitting is stable under $Cl_{0}(4,C)$:

$\forall w\in Cl_{0}(4,C):\vartheta(w,g)\psi=\sum_{k,l}\psi^{kl}[\rho\circ\Upsilon(w)]_{k}^{i}[\chi(g)]_{l}^{j}e_{i}\otimes f_{j}$

$=\sum_{k,l,p}[\gamma_{+}]_{p}^{k}\psi^{pl}[\rho\circ\Upsilon(w)]_{k}^{i}[\chi(g)]_{l}^{j}e_{i}\otimes f_{j}$

$+\sum_{k,l}[\gamma_{-}]_{p}^{k}\psi^{pl}[\rho\circ\Upsilon(w)]_{k}^{i}[\chi(g)]_{l}^{j}e_{i}\otimes f_{j}$

$=\sum_{k,l,p}\psi^{pl}[\rho\circ\Upsilon(w)\circ\gamma_{+}]_{p}^{i}[\chi(g)]_{l}^{j}e_{i}\otimes f_{j}$

$+\sum_{k,l}\psi^{pl}[\rho\circ\Upsilon(w)\circ\gamma_{-}]_{p}^{i}[\chi(g)]_{l}^{j}e_{i}\otimes f_{j}$

$=\sum_{k,l,p}\psi^{pl}[\gamma_{+}\circ\rho\circ\Upsilon(w)]_{p}^{i}[\chi(g)]_{l}^{j}e_{i}\otimes f_{j}$

$+\sum_{k,l}\psi^{pl}[\gamma_{-}\circ\rho\circ\Upsilon(w)]_{p}^{i}[\chi(g)]_{l}^{j}e_{i}\otimes f_{j}$

$=\sum_{k,l,p}[\gamma_{+}]_{k}^{i}[\rho\circ\Upsilon(w)]_{p}^{k}[\chi(g)]_{l}^{j}\psi^{pl}e_{i}\otimes f_{j}$

$+\sum_{k,l}[\gamma_{-}]_{k}^{i}[\rho\circ\Upsilon(w)]_{p}^{k}[\chi(g)]_{l}^{j}\psi^{pl}e_{i}\otimes f_{j}\blacksquare$

The two vector spaces are isotropic for the scalar product defined previously :

$\langle\psi_{1},\psi_{2}\rangle=Tr([\psi_{1}]^{*}\gamma_{0}[\psi_{2}])=\sum_{ijk}\gamma_{4ij}\overline{\psi}_{1}^{ik}\psi_{2}^{jk}$

If $[\psi_{2}]=\gamma_{\epsilon}\psi_{2},[\psi_{1}]=\gamma_{\epsilon'}\psi_{1}:[\psi_{1}]^{*}\gamma_{0}[\psi_{2}]=[\psi_{1}]^{*}\gamma_{\epsilon}\gamma_{0}\gamma_{\epsilon'}[\psi_{2}]=[\psi_{1}]^{*}\gamma_{\epsilon}\gamma_{-\epsilon'}\gamma_{0}[\psi_{2}]$

$\varepsilon=\varepsilon'\Rightarrow\langle\psi_{1},\psi_{2}\rangle=0\blacksquare$

5) ε_{5} does not depend of the choice of an orthonormal basis, and so for γ_{ϵ}.The splitting can be lifted on the fiber bundle. At each m the fiber over m can be splitted in $F^{+}\otimes W+F^{-}\otimes W$.

13.2 A choice of the F space

1) These features lead to precise the choice of a specific basis of the vector space F : a basis which reflects the splitting in the direct sum of two 2 dimensional complex vector spaces.

Let be this basis such that F^+ corresponds to the two first vectors (e_1, e_2) and F^- to the last (e_3, e_4) :

$$F^+ = \left\{ \begin{bmatrix} \phi_1 \\ \phi_2 \\ 0 \\ 0 \end{bmatrix} \right\}, F^- = \left\{ \begin{bmatrix} 0 \\ 0 \\ \phi_3 \\ \phi_4 \end{bmatrix} \right\}$$

The conditions : $F^+ = \gamma_+ F, F^- = \gamma_- F; \gamma_+ + \gamma_- = I; \gamma_\epsilon \gamma_{-\epsilon} = 0$ lead to

$$\gamma_+ = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}, \gamma_- = \begin{bmatrix} 0 & 0 \\ 0 & I_2 \end{bmatrix}$$

Thus $\gamma_5 = \gamma_+ - \gamma_- = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$

The γ_k matrices must meet : $\gamma_i \gamma_j + \gamma_j \gamma_i = 2\delta_{ij} I_4$ and we assume as before that : $\gamma_k = \gamma_k^* \Rightarrow \gamma_k \gamma_k^* = I$

So $\gamma_k = \begin{bmatrix} A_k & B_k \\ B_k^* & D_k \end{bmatrix}$

with $A_k = A_k^*, D_k = D_k^*, A_k^2 + B_k B_k^* = I = B_k^* B_k + D_k^2, A_k B_k + B_k D_k = 0$

The conditions k=0..3: $\gamma_5 \gamma_k = -\gamma_k \gamma_5$ impose : $A_k = D_k = 0$

so $\gamma_k = \begin{bmatrix} 0 & B_k \\ B_k^* & 0 \end{bmatrix}$ with $B_k B_k^* = I_2$

The condition $\gamma_5 = \gamma_0 \gamma_1 \gamma_2 \gamma_3$ imposes :$B_0 B_1^* B_2 B_3^* = I_2, B_1 B_0^* B_3 B_2^* = -I_2$

2) There are not many choices left, and we come to the solution :

$$\gamma_1 = \begin{bmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{bmatrix}; \gamma_2 = \begin{bmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{bmatrix}; \gamma_3 = \begin{bmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{bmatrix}; \gamma_0 = i \begin{bmatrix} 0 & \sigma_0 \\ -\sigma_0 & 0 \end{bmatrix}$$

with the Pauli matrices :

$$\sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \sigma_2 = i \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}; \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix};$$

$i, j = 1, 2, 3 : \sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{ij} \sigma_0; \sigma_i = \sigma_i^*; \sigma_1 \sigma_2 = i\sigma_3; \sigma_2 \sigma_3 = i\sigma_1; \sigma_3 \sigma_1 = i\sigma_2$

With this choice we have :

$$\gamma^r = \begin{bmatrix} 0 & \sigma_r \\ \eta^{rr} \sigma_r & 0 \end{bmatrix}$$

a<4 : $[\kappa_a] = -i\frac{1}{2} \begin{bmatrix} \sigma_a & 0 \\ 0 & \sigma_a \end{bmatrix}$; a>3 : $[\kappa_a] = \frac{1}{2} \begin{bmatrix} \sigma_{a-3} & 0 \\ 0 & -\sigma_{a-3} \end{bmatrix}$

3) The kinematic part of the state of particles is described in a two 2-complex components vector : $\phi = \begin{bmatrix} \phi_R \\ \phi_L \end{bmatrix}$, and $\phi_R \in F^+, \phi_L \in F^-$ are Weyl spinors. The state is described in a section $\psi = \begin{bmatrix} \psi_R \\ \psi_L \end{bmatrix}$ where $\psi_R = \psi_R^{ij}, \psi_L = \psi_L^{ij}, j = 1..m, i = 1, 2$

We will denote $[\psi_R], [\psi_L]$ the 2xm matrices.

14 LAGRANGIAN

The lagrangian is comprised of 3 parts, related to the particles alone, the field forces alone, and the interactions.

14.1 Particules

1) For the particles the lagrangian cannot depend of the derivatives, which depend on the fields. So the simplest choice is :

$a_M N \langle \psi, \psi \rangle = a_M N Tr\left([\psi]^* \gamma_0 [\psi]\right)$ with some real scalar a_M

$$\langle \psi, \psi \rangle = Tr\left(\begin{bmatrix} \psi_R^* & \psi_L^* \end{bmatrix} i \begin{bmatrix} 0 & \sigma_0 \\ -\sigma_0 & 0 \end{bmatrix} \begin{bmatrix} \psi_R \\ \psi_L \end{bmatrix} \right) = iTr\left([\psi_R]^* [\psi_L] - [\psi_L]^* [\psi_R]\right)$$

$$\overline{Tr\left([\psi_R]^* [\psi_L]\right)} = Tr\left(\left([\psi_R]^* [\psi_L]\right)^*\right) = Tr\left([\psi_L]^* [\psi_R]\right)$$

$$Tr\left([\psi_R]^* [\psi_L] - [\psi_L]^* [\psi_R]\right) = 2i \operatorname{Im} Tr\left([\psi_R]^* [\psi_L]\right)$$

$$\langle \psi, \psi \rangle = -2 \operatorname{Im} Tr\left([\psi_R]^* [\psi_L]\right)$$

This quantity does not depend of the jauge or the chart : this a function on Ω, evaluated at $\widetilde{f}(m)$.

2) It would be legitimate to add some dynamic part like $\sum_\alpha V^\alpha V_\alpha$ but it raises two issues. First we should put some "mass",which figures in ψ. Second the mass is the "charge" associated to the gravitation field, so it makes sense to keep the dynamic part linked with the covariant derivative $\nabla \psi$. We will see how.

14.2 Fields

1) The lagrangian depends on the curvature forms and cannot involve ψ or explicitely $\grave{A}$. We will assume that the derivatives $\partial_\alpha O'$ are not present, thus

G does not appear and the lagrangian depends uniquely on the curvature forms.

2) The simplest solution is to take the scalar product defined before. For $\mathcal{F}_A$ it reads :

$a_F \langle \mathcal{F}_A, \mathcal{F}_A \rangle = a_F \sum_a \sum_{\{\alpha\beta\}} \overline{\mathcal{F}}^a_{A\alpha\beta} \mathcal{F}_A^{a,\alpha\beta}$

$= a_F \sum_a \sum_{\{\alpha\beta\}} \overline{\mathcal{F}}^a_{A\alpha\beta} \sum_{\lambda\mu} g^{\alpha\lambda} g^{\beta\mu} \mathcal{F}^a_{A\lambda\mu} = a_F \frac{1}{2} \sum_a \sum_{\alpha\beta\lambda\mu} g^{\alpha\lambda} g^{\beta\mu} \overline{\mathcal{F}}^a_{A\alpha\beta} \mathcal{F}^a_{A\lambda\mu}$

This a real quantity because this scalar product is hermitian.

3) The same choice for gravitation would lead to $a_G \sum_{\{\alpha\beta\}} B\left(\mathcal{F}_{G\alpha\beta}, \mathcal{F}_G^{\alpha\beta}\right)$ with some scalar product on o(3,1). The only natural choice for this scalar product is the Killing form $B(X,Y) = 2Tr(\left[\widetilde{X}\right]\left[\widetilde{Y}\right])$ which gives with the Riemann tensor :

$\frac{1}{4} a_G \sum_{\alpha\beta\lambda\mu} g^{\alpha\lambda} g^{\beta\mu} B\left(\mathcal{F}_{B\alpha\beta}, \mathcal{F}_{B\lambda\mu}\right)$

$= \frac{1}{2} a_G \sum_{\alpha\beta\lambda\mu} g^{\alpha\lambda} g^{\beta\mu} Tr\left(\left[\widetilde{\mathcal{F}}_{B\alpha\beta}\right]\left[\widetilde{\mathcal{F}}_{B\lambda\mu}\right]\right)$

$= \frac{1}{2} a_G \sum_{\alpha\beta\lambda\mu} g^{\alpha\lambda} g^{\beta\mu} Tr\left(O^{-1}\left[R_{\alpha\beta}\right] O O^{-1}\left[R_{\lambda\mu}\right] O\right)$

$= \frac{1}{2} a_G \sum_{\alpha\beta\lambda\mu} g^{\alpha\lambda} g^{\beta\mu} Tr\left(\left[R_{\alpha\beta}\right]\left[R_{\lambda\mu}\right]\right)$

$= \frac{1}{2} a_G \sum_{\alpha\beta\lambda\mu} g^{\alpha\lambda} g^{\beta\mu} \sum_{cd} R^c_{\alpha\beta d} R^d_{\lambda\mu c}$

$= \frac{1}{2} a_G \sum_{\alpha\beta\lambda\mu} g^{\alpha\lambda} g^{\beta\mu} Tr\left(\left[R_{\alpha\beta}\right]\left[R_{\lambda\mu}\right]\right)$

The trouble is that the Killing form is not positive definite. It is possible to turn over this problem by using the "compact real form" of o(3,1) (Knapp [12] VI.1), in fact treating gravitation on the same footing as the other fields, and considering a complex valued connection. But this would move us further away from the traditional affine connections.

The alternate option is use the Palatini action, as in General Relativity. It can be computed with our variables, as seen before :

The scalar curvature is (equation 11):

$R = \sum_{\alpha\beta ijk} \left[\widetilde{\mathcal{F}_{G\alpha\beta}}\right]^i_k \eta^{kj} O^\beta_i O^\alpha_j = \sum_{a,\alpha\beta} \mathcal{F}^a_{G\alpha\beta}\left(O^\beta_{p_a} O^\alpha_{q_a} - O^\beta_{q_a} O^\alpha_{p_a}\right)$

and the lagrangian is $a_G \sum_{a\alpha\beta ij} \mathcal{F}^a_{G\alpha\beta}\left(\left[\widetilde{\kappa}_a\right][\eta]\right)^i_j O^\beta_i O^\alpha_j$ with a real scalar a_G.

So explicitly :

$L_F = a_F \langle \mathcal{F}_A, \mathcal{F}_A \rangle + a_G \sum_{a,\alpha\beta} \mathcal{F}^a_{G\alpha\beta}\left(O^\beta_{p_a} O^\alpha_{q_a} - O^\beta_{q_a} O^\alpha_{p_a}\right)$

$L_F = \frac{1}{2} a_F \sum_{a\alpha\beta\lambda\mu ijkl} \eta^{ji} \eta^{lk} O^\lambda_j O^\alpha_i O^\mu_l O^\beta_k \mathcal{F}^a_{A\alpha\beta} \overline{\mathcal{F}}^a_{A\lambda\mu}$

$\quad + a_G \sum_{a,\alpha\beta} \mathcal{F}^a_{G\alpha\beta}\left(O^\beta_{p_a} O^\alpha_{q_a} - O^\beta_{q_a} O^\alpha_{p_a}\right)$

Remark : one could introduce a cosmological constant Λ such that the lagrangian becomes : $a_G(\Lambda + R)$ and Λ acts in the action through the density $\sqrt{|\det g|}$.

14.3 Interactions

As said before we have to address two interactions.

1) The "static" part : It models the pure action on the state of particles. This is for what we defined the Dirac operator. So the simplest choices are : $\langle D\psi, D\psi\rangle, \langle D\psi, \psi\rangle, \langle \psi, D\psi\rangle$. The first one leads to quadratic terms in the derivatives, but the two last ones cannot guarantee to deliver a real scalar. So we have two options :

$\frac{1}{2}(\langle D\psi, \psi\rangle + \langle \psi, D\psi\rangle) = \operatorname{Re}\langle \psi, D\psi\rangle$

$\frac{1}{2}i(\langle D\psi, \psi\rangle - \langle \psi, D\psi\rangle) = \operatorname{Im}\langle \psi, D\psi\rangle$

Clearly the term in $\partial_\alpha\psi$ in the lagrangian is crucial. That is

$\sum_r Tr([\psi^*]\gamma_0\gamma^r[\partial_\alpha\psi]) = iTr(\eta^{rr}[\psi_R]^*\sigma_r[\partial_r\psi_R] - [\psi_L]^*\sigma_r[\partial_r\psi_L])$

with $\operatorname{Re}\langle \psi, D\psi\rangle$ we have

$\sum_r \operatorname{Re} Tr([\psi^*]\gamma_0\gamma^r[\partial_\alpha\psi]) = \sum_r\{-\operatorname{Im} Tr(\eta^{rr}[\psi_R]^*\sigma_r[\partial_r\psi_R] - [\psi_L]^*\sigma_r[\partial_r\psi_L])$

with $\operatorname{Im}\langle \psi, D\psi\rangle$ we have

$\sum_r \operatorname{Im} Tr([\psi^*]\gamma_0\gamma^r[\partial_\alpha\psi]) = \sum_r\{\operatorname{Re} Tr(\eta^{rr}[\psi_R]^*\sigma_r[\partial_r\psi_R] - [\psi_L]^*\sigma_r[\partial_r\psi_L])$

and we have the identities : $\operatorname{Re} Tr([\psi]^*\sigma_r[\partial_\alpha\psi]) = \frac{1}{2}\partial_\alpha Tr([\psi]^*\sigma_r[\psi])$

Indeed :

$Tr[\psi]^*\sigma_r[\partial_r\psi]$

$= \sum_\alpha O'^{\alpha}_r \sum_{klj}\sigma_r^{kl}\overline{\psi}^{kj}\partial_\alpha\psi^{lj}$

$= \sum_\alpha O'^{\alpha}_r \sum_j \sigma_r^{11}\overline{\psi}^{1j}\partial_\alpha\psi^{1j} + \sigma_r^{12}\overline{\psi}^{1j}\partial_\alpha\psi^{2j} + \sigma_r^{21}\overline{\psi}^{2j}\partial_\alpha\psi^{1j} + \sigma_r^{22}\overline{\psi}^{2j}\partial_\alpha\psi^{2j}$

$Tr[\psi]^*\sigma_0[\partial_r\psi] = \frac{1}{2}\partial_r\sum_{ij}|\psi^{ij}|^2$

$Tr[\psi]^*\sigma_1[\partial_r\psi] = \sum_j \partial_r(\psi^{1j}\psi^{2j})$

$Tr[\psi]^*\sigma_2[\partial_r\psi] = i\sum_j -\overline{\psi}^{1j}\partial_r\psi^{2j} + \overline{\psi}^{2j}\partial_r\psi^{1j}$

$Tr[\psi]^*\sigma_3[\partial_r\psi] = \frac{1}{2}\sum_j \partial_r\left(|\psi^{1j}|^2 - |\psi^{2j}|^2\right)$

As we see if we take the imaginary part of the above expressions il would be null for $r = 0, 3$ thus making two privileged directions, and that does not happen with the real part. It is a rather weak argument, but let us say that this option has been tested before (Giachetta [5]).

So we choose $\epsilon = -1$ and the following lagrangian :

$a_I\frac{1}{2}i(\langle D\psi, \psi\rangle - \langle \psi, D\psi\rangle) = a_I \operatorname{Im}\langle \psi, D\psi\rangle$

2) The "dynamic part" It involves the fields, the state of particles and their velocity. The simplest choice is :

$\langle \psi, \sum_\alpha V^\alpha \nabla_\alpha \psi \rangle$

V is a vector field, so this quantity is covariant, and invariant in a change of gauge. As above we need a real quantity. There is no obvious reason for one or the other option. Let us say that after testing both, the most physically meaningful is the same as above. So I take :

$a_D \sum_\alpha \frac{1}{2} V^\alpha i \left(\langle \nabla_\alpha \psi, \psi \rangle + \langle \psi, \nabla_\alpha \psi \rangle \right) = a_D \sum_\alpha V^\alpha \operatorname{Im} \langle \psi, \nabla_\alpha \psi \rangle$

with a real scalar constant a_D

14.4 Summary

The full lagrangian of this model is :

$$(79)$$

$$\mathcal{L}_M = N \left(a_M \langle \psi, \psi \rangle + a_I \operatorname{Im} \langle \psi, D\psi \rangle + a_D \sum_\alpha V^\alpha \operatorname{Im} \langle \psi, \nabla_\alpha \psi \rangle \right) \det O'$$
$$\mathcal{L}_F = \left(a_F \langle \mathcal{F}_A, \mathcal{F}_A \rangle + a_G R \right) \det O'$$

The lagrangian is defined through intrinsic quantities so it is invariant under gauge transformations or change of charts.

It does not involve the partial derivatives $\partial_\alpha O'^i_\beta$ of the tetrad. There is no obvious need to introduce them, as the scalar curvature answers to the interaction between the geometry and the gravitational field (as it appears very clearly in the formula $R = \sum_{ijk} \left[F \left(\partial_p, \partial_q \right) \right]^q_j \eta^{jp}$) . An additional item should be purely geometrical in nature. In some ways a cosmological constant (which acts through the volume form) is a tentative solution, but "purely scalar" in that it does not involve any space-time distorsion other than dilation. Eventually what is missing is some equivalent of the term $a_M \langle \psi, \psi \rangle$, meaning that space-time itself is more than a container, and possesses some intrinsic property independantly from matter and force fields, all this without reinventing the aether...So for the simple purpose I have in mind I will keep the simplest solution and stick to this basic lagrangian.

The matter part of the lagrangian writes :

$\mathcal{L}_M = N \left(\det O' \right) \left(a_M \langle \psi, \psi \rangle + a_I \operatorname{Im} \langle \psi, D\psi \rangle + a_D \sum_\alpha V^\alpha \operatorname{Im} \langle \psi, \nabla_\alpha \psi \rangle \right)$

$= N \left(\det O' \right) \left(a_M \langle \psi, \psi \rangle + \operatorname{Im} \langle \psi, \sum_r a_I O^\alpha_r \left[\gamma^r \right] \left[\nabla_\alpha \psi \right] \rangle + \operatorname{Im} \langle \psi, \sum_\alpha V^\alpha a_D \nabla_\alpha \psi \rangle \right)$

$= N \left(\det O' \right) \left(a_M \langle \psi, \psi \rangle + \operatorname{Im} \langle \psi, \left(\sum_r a_I O^\alpha_r \left[\gamma^r \right] + \sum_\alpha V^\alpha a_D I \right) \left[\nabla_\alpha \psi \right] \rangle \right)$

with I the unitary 4x4 matrix, and it will be convenient to denote the operator :

$$D_M^\alpha = \left(\sum_r a_I O_r^\alpha [\gamma^r] + \sum_\alpha V^\alpha a_D I \right) \tag{80}$$

so : $\mathcal{L}_M = N (\det O') \left(a_M \langle \psi, \psi \rangle + \operatorname{Im} \langle \psi, \sum_\alpha D_M^\alpha [\nabla_\alpha \psi] \rangle \right)$

With this notation it is obvious that $a_I \operatorname{Im} \langle \psi, D\psi \rangle$ accounts for the kinematic part (rotations *in* the tetrad) and $a_D \sum_\alpha V^\alpha \operatorname{Im} \langle \psi, \nabla_\alpha \psi \rangle$ for the dynamic part (displacement within Ω).

15 LAGRANGE EQUATIONS

The Lagrange equations are just the transcription of the previous ones but, as expected, one can get more insightful results.

15.1 Gravitation

We will define two moments : P, which can be seen as a "linear momentum", and J, as an "angular momentum", both computed from the state tensor, but gauge and chart invariant. The most important result is that the gravitational potential G can be explicitly computed from the structure coefficients and these two moments. The torsion is then easily computed and it appears that usually the gravitational connection would not be torsionfree.

15.1.1 Noether currents

1) The Noether current reads (equation 58):

$Y_G = \sum_{a,\alpha} \frac{d(VL_M + L_F)}{dG_\alpha^a} \partial_\alpha \otimes \overrightarrow{\kappa}_a$

with $Y_G^{a\alpha} = V \sum_{ij} \left(\frac{dL_M^\Diamond}{d \operatorname{Re} \nabla_\alpha \psi^{ij}} \operatorname{Re} ([\kappa_a] [\psi])_j^i + \frac{dL_M^\Diamond}{d \operatorname{Im} \nabla_\alpha \psi^{ij}} \operatorname{Im} ([\kappa_a] [\psi])_j^i \right)$

$+V \frac{\partial L_M^\Diamond}{\partial G_\alpha^a} + \frac{\partial L_F}{\partial G_\alpha^a} + 2 \sum_{b,\beta} \frac{dL_F}{d\mathcal{F}_{G\alpha\beta}^b} [\overrightarrow{\kappa}_a, G_\beta]^b$

$V \frac{\partial L_M^\Diamond}{\partial G_\alpha^a} = \frac{\partial L_F}{\partial G_\alpha^a} = 0$

$\frac{dL_F}{d\mathcal{F}_{G\alpha\beta}^b} = a_G \left(O_{p_b}^\beta O_{q_b}^\alpha - O_{q_b}^\beta O_{p_b}^\alpha \right)$

Let us compute the partial derivatives with respect to $\operatorname{Re} \nabla_\alpha \psi^{ij}, \operatorname{Im} \nabla_\alpha \psi^{ij}$:

$L_M = N\left(a_M\left\langle\psi,\psi\right\rangle + \operatorname{Im}\left\langle\psi,\sum_\alpha D_M^\alpha\left[\nabla_\alpha\psi\right]\right\rangle\right)$
$\operatorname{Im}\left\langle\psi,\sum_\alpha D_M^\alpha\left[\nabla_\alpha\psi\right]\right\rangle$
$= \sum_{\alpha pq}\operatorname{Im}\left(\left[\psi\right]^*\left[\gamma_0\right]\left[D_M^\alpha\right]\right)_q^p\left[\nabla_\alpha\psi\right]^{qp}$
$= \sum_{\alpha pq}\operatorname{Re}\left(\left[\psi\right]^*\left[\gamma_0\right]\left[D_M^\alpha\right]\right)_q^p\operatorname{Im}\left[\nabla_\alpha\psi\right]^{qp} + \operatorname{Im}\left(\left[\psi\right]^*\left[\gamma_0\right]\left[D_M^\alpha\right]\right)_q^p\operatorname{Re}\left[\nabla_\alpha\psi\right]^{qp}$
$\frac{d\operatorname{Im}\left\langle\psi,\sum_\alpha D_M^\alpha\left[\nabla_\alpha\psi\right]\right\rangle}{d\operatorname{Re}\nabla_\alpha\psi^{ij}} = \sum_{pq}\operatorname{Im}\left(\left[\psi\right]^*\left[\gamma_0\right]\left[D_M^\alpha\right]\right)_q^p\delta_q^i\delta_p^j = \operatorname{Im}\left(\left[\psi\right]^*\left[\gamma_0\right]\left[D_M^\alpha\right]\right)_i^j$
$\frac{d\operatorname{Im}\left\langle\psi,\sum_\alpha D_M^\alpha\left[\nabla_\alpha\psi\right]\right\rangle}{d\operatorname{Im}\nabla_\alpha\psi^{ij}} = \sum_{pq}\operatorname{Re}\left(\left[\psi\right]^*\left[\gamma_0\right]\left[D_M^\alpha\right]\right)_q^p\delta_q^i\delta_p^j = \operatorname{Re}\left(\left[\psi\right]^*\left[\gamma_0\right]\left[D_M^\alpha\right]\right)_i^j$
$\frac{dL_M}{d\operatorname{Re}\nabla_\alpha\psi^{ij}} = N\operatorname{Im}\left(\left[\psi\right]^*\left[\gamma_0\right]\left[D_M^\alpha\right]\right)_i^j$
$\frac{dL_M}{d\operatorname{Im}\nabla_\alpha\psi^{ij}} = N\operatorname{Re}\left(\left[\psi\right]^*\left[\gamma_0\right]\left[D_M^\alpha\right]\right)_i^j$
So :
$Y_G^{a\alpha} = N\sum_{ij}\left(\operatorname{Im}\left(\left[\psi\right]^*\left[\gamma_0\right]\left[D_M^\alpha\right]\right)_i^j\operatorname{Re}\left(\left[\kappa_a\right]\left[\psi\right]\right)_j^i + \operatorname{Re}\left(\left[\psi\right]^*\left[D_M^\alpha\right]\right)_i^j\operatorname{Im}\left(\left[\kappa_a\right]\left[\psi\right]\right)_j^i\right)$
$+2\sum_{b,\beta}a_G\left(O_{p_b}^\beta O_{q_b}^\alpha - O_{q_b}^\beta O_{p_b}^\alpha\right)\left[\overrightarrow{\kappa}_a,G_\beta\right]^b$
$Y_G^{a\alpha} = N\operatorname{Im}Tr\left[\psi\right]^*\left[\gamma_0\right]\left[D_M^\alpha\right]\left(\left[\kappa_a\right]\left[\psi\right]\right)+2\sum_{b,\beta}a_G\left(O_{p_b}^\beta O_{q_b}^\alpha - O_{q_b}^\beta O_{p_b}^\alpha\right)\left[\overrightarrow{\kappa}_a,G_\beta\right]^b$
In the orthonormal basis :
$Y_G = \left(\sum_{a,r}N\operatorname{Im}Tr\left[\psi\right]^*\left[D_M^r\right]\left(\left[\kappa_a\right]\left[\psi\right]\right)\right.$
$\left.+2a_G\sum_{b,\alpha\beta}\left(O_{p_b}^\beta O_{q_b}^\alpha - O_{q_b}^\beta O_{p_b}^\alpha\right)\left[\overrightarrow{\kappa}_a,G_\beta\right]^b O_\alpha^{\prime r}\right)\partial_r\otimes\overrightarrow{\kappa}_a$
with $\left[D_M^r\right] = D_M^\alpha = \left(a_I\left[\gamma^r\right] + \sum_{r\alpha}O_\alpha^{\prime r}V^\alpha a_D I\right) = \left(a_I\left[\gamma^r\right] + V^r a_D I\right)$
$Y_G = \sum_{a,r}\{N\left(a_I\left(\operatorname{Im}Tr\left(\left[\psi\right]^*\left[\gamma_0\gamma^r\right]\left[\kappa_a\right]\left[\psi\right]\right) + a_D V^r\operatorname{Im}Tr\left(\left[\psi^*\right]\left[\gamma_0\right]\left[\kappa_a\right]\left[\psi\right]\right)\right)\right)$
$+2a_G\sum_{b,\alpha\beta}\left(O_{p_b}^\beta O_{q_b}^\alpha - O_{q_b}^\beta O_{p_b}^\alpha\right)\left[\overrightarrow{\kappa}_a,G_\beta\right]^b O_\alpha^{\prime r}\}\partial_r\otimes\overrightarrow{\kappa}_a$

2) We have for the first term: $a_I\operatorname{Im}Tr\left(\left[\psi\right]^*\left[\gamma_0\gamma^r\right]\left[\kappa_a\right]\left[\psi\right]\right)$
a<4 : $Tr\left(\left[\psi\right]^*\left[\gamma_0\gamma^r\right]\left[\kappa_a\right]\left[\psi\right]\right) = \frac{1}{2}Tr\left(\eta^{rr}\psi_R^*\sigma_r\sigma_a\psi_R - \psi_L^*\sigma_r\sigma_a\psi_L\right)$
a>3 : $Tr\left(\left[\psi\right]^*\left[\gamma_0\gamma^r\right]\left[\kappa_a\right]\left[\psi\right]\right) = i\frac{1}{2}Tr\left(\eta^{rr}\psi_R^*\sigma_r\sigma_{a-3}\psi_R + \psi_L^*\sigma_r\sigma_{a-3}\psi_L\right)$
with $\sigma_1\sigma_2 = i\sigma_3; \sigma_2\sigma_3 = i\sigma_1; \sigma_3\sigma_1 = i\sigma_2$ the quantity $Tr\left(\left[\psi\right]^*\left[\gamma_0\gamma^r\right]\left[\kappa_a\right]\left[\psi\right]\right)$ is given by the table :

$$\begin{bmatrix} a\backslash r & 0 & 1 \\ 1 & -\frac{1}{2}Tr\left(\psi_R^*\sigma_1\psi_R + \psi_L^*\sigma_1\psi_L\right) & \frac{1}{2}Tr\left(\psi_R^*\psi_R - \psi_L^*\psi_L\right) \\ 2 & -\frac{1}{2}Tr\left(\psi_R^*\sigma_2\psi_R + \psi_L^*\sigma_2\psi_L\right) & i\frac{1}{2}Tr\left(\psi_R^*\sigma_3\psi_R - \psi_L^*\sigma_3\psi_L\right) \\ 3 & -\frac{1}{2}Tr\left(\psi_R^*\sigma_3\psi_R + \psi_L^*\sigma_3\psi_L\right) & -i\frac{1}{2}Tr\left(\psi_R^*\sigma_2\psi_R - \psi_L^*\sigma_2\psi_L\right) \\ 4 & -i\frac{1}{2}Tr\left(\psi_R^*\sigma_1\psi_R - \psi_L^*\sigma_1\psi_L\right) & i\frac{1}{2}Tr\left(\psi_R^*\psi_R + \psi_L^*\psi_L\right) \\ 5 & -i\frac{1}{2}Tr\left(\psi_R^*\sigma_2\psi_R - \psi_L^*\sigma_2\psi_L\right) & -\frac{1}{2}Tr\left(\psi_R^*\sigma_3\psi_R + \psi_L^*\sigma_3\psi_L\right) \\ 6 & -i\frac{1}{2}Tr\left(\psi_R^*\sigma_3\psi_R - \psi_L^*\sigma_3\psi_L\right) & \frac{1}{2}Tr\left(\psi_R^*\sigma_2\psi_R + \psi_L^*\sigma_2\psi_L\right) \end{bmatrix}$$

$$\begin{bmatrix} a\backslash r & 2 & 3 \\ 1 & -i\frac{1}{2}Tr\left(\psi_R^*\sigma_3\psi_R-\psi_L^*\sigma_3\psi_L\right) & i\frac{1}{2}Tr\left(\psi_R^*\sigma_2\psi_R-\psi_L^*\sigma_2\psi_L\right) \\ 2 & \frac{1}{2}Tr\left(\psi_R^*\psi_R-\psi_L^*\psi_L\right) & -i\frac{1}{2}Tr\left(\psi_R^*\sigma_1\psi_R-\psi_L^*\sigma_1\psi_L\right) \\ 3 & i\frac{1}{2}Tr\left(\psi_R^*\sigma_1\psi_R-\psi_L^*\sigma_1\psi_L\right) & \frac{1}{2}Tr\left(\psi_R^*\psi_R-\psi_L^*\psi_L\right) \\ 4 & \frac{1}{2}Tr\left(\psi_R^*\sigma_3\psi_R+\psi_L^*\sigma_3\psi_L\right) & -\frac{1}{2}Tr\left(\psi_R^*\sigma_2\psi_R+\psi_L^*\sigma_2\psi_L\right) \\ 5 & i\frac{1}{2}Tr\left(\psi_R^*\psi_R+\psi_L^*\psi_L\right) & \frac{1}{2}Tr\left(\psi_R^*\sigma_1\psi_R+\psi_L^*\sigma_1\psi_L\right) \\ 6 & -\frac{1}{2}Tr\left(\psi_R^*\sigma_1\psi_R+\psi_L^*\sigma_1\psi_L\right) & i\frac{1}{2}Tr\left(\psi_R^*\psi_R+\psi_L^*\psi_L\right) \end{bmatrix}$$

As $Tr\left(\psi^*\sigma_k\psi\right)\in\mathbb{R}$ the quantity $\operatorname{Im}Tr\left([\psi]^*[\gamma_0\gamma^r][\kappa_a][\psi]\right)$ is given by the table :

$$\operatorname{Im}Tr\left([\psi]^*[\gamma_0\gamma^r][\kappa_a][\psi]\right)=\begin{bmatrix} a\backslash r & 0 & 1 & 2 & 3 \\ 1 & 0 & 0 & J^3 & -J^2 \\ 2 & 0 & -J^3 & 0 & J^1 \\ 3 & 0 & J^2 & -J^1 & 0 \\ 4 & J^1 & J^0 & 0 & 0 \\ 5 & J^2 & 0 & J^0 & 0 \\ 6 & J^3 & 0 & 0 & J^0 \end{bmatrix}$$

with

$$\mathbf{J}_k=-\frac{1}{2}\mathbf{Tr}\left(\eta^{kk}\psi_R^*\sigma_k\psi_R-\psi_L^*\sigma_k\psi_L\right) \tag{81}$$

One can check that :
$\operatorname{Im}Tr\left([\psi]^*[\gamma_0\gamma^r][\kappa_a][\psi]\right)=\sum_k J_k\left[\widetilde{\kappa}_a\right]_r^k$
We denote by $[J]$ the 1x4 row matrix J_k. So :

$$\operatorname{Im}\langle\psi,[\gamma^r][\kappa_a][\psi]\rangle=\operatorname{Im}\mathbf{Tr}\left([\psi]^*[\gamma_0\gamma^r][\kappa_a][\psi]\right)=\left([J]\left[\widetilde{\kappa}_a\right]\right)_r \tag{82}$$

and :
$\sum_{a,r}\operatorname{Im}Tr\left([\psi]^*[\gamma_0\gamma^r][\kappa_a][\psi]\right)\partial_r\otimes\overrightarrow{\kappa}_a$
$=\sum_{a,r}\left(J_{p_a}\partial_{q_a}-\eta^{p_ap_a}J_{q_a}\partial_{p_a}\right)\otimes\overrightarrow{\kappa}_a=\sum_{a,r}\left([J]\left[\widetilde{\kappa}_a\right]\right)_r\partial_r\otimes\overrightarrow{\kappa}_a$

The quantities J_k are real scalar functions, invariant in a gauge transformations, and so are not components of a vector field on M. We see that the geometrical pertinent quantity is $\sum_{a,r}\left([J]\left[\widetilde{\kappa}_a\right]\right)_r\partial_r$ which is a vector, similar to a relativistic angular momentum.

Remark : one can check :
$[\psi]^*[\gamma_0\gamma^r][\psi]=i\left(\eta^{kk}\psi_R^*\sigma_k\psi_R-\psi_L^*\sigma_k\psi_L\right)$
So $\langle\psi,\gamma^r\psi\rangle=Tr\left([\psi]^*[\gamma_0\gamma^r][\psi]\right)=iTr\left(\eta^{kk}\psi_R^*\sigma_k\psi_R-\psi_L^*\sigma_k\psi_L\right)=-2J_r$

3) The second item : $a_D V^r \operatorname{Im} Tr\left([\psi^*][\gamma_0][\kappa_a][\psi]\right)$

$\underline{Tr\left([\psi^*][\gamma_0][\kappa_a][\psi]\right)}$ is a pure imaginary scalar as one can check :

$\overline{Tr\left([\psi^*][\gamma_0][\kappa_a][\psi]\right)} = Tr\left([\psi^*][\gamma_0][\kappa_a][\psi]\right)^*$

$= Tr\left([\psi^*][\kappa_a]^*[\gamma_0][\psi]\right) = -Tr\left([\psi^*][\gamma_0][\kappa_a][\gamma_0][\gamma_0][\psi]\right) = -Tr\left([\psi^*][\gamma_0][\kappa_a][\psi]\right)$

So let be $P_a \in \mathbb{R} : Tr\left([\psi^*][\gamma_0][\kappa_a][\psi]\right) = iP_a$

a<4 : $Tr\left([\psi^*][\gamma_0][\kappa_a][\psi]\right) = \frac{1}{2} Tr\left([\psi_R^*][\sigma_a][\psi_L] - [\psi_L^*][\sigma_a][\psi_R]\right)$

a>3 :$Tr\left([\psi^*][\gamma_0][\kappa_a][\psi]\right) = -i\frac{1}{2} Tr\left([\psi_R^*][\sigma_{a-3}][\psi_L] + [\psi_L^*][\sigma_{a-3}][\psi_R]\right)$

So :

$$\langle \psi, [\kappa_a][\psi]\rangle = \mathbf{Tr}\left([\psi^*][\gamma_0][\kappa_a][\psi]\right) = \mathbf{iP}_a \tag{83}$$

The quantities P_a are real scalar functions, invariant in a gauge transformations. The geometrical pertinent quantity is $\sum_{a,r} V^r P_a \partial_r \otimes \overrightarrow{\kappa}_a$.

One can check that :

$\sum_{a=1}^6 (P_a)^2$

$= \sum_{a=1}^3 \left(-i\frac{1}{2} Tr\left([\psi_R^*][\sigma_a][\psi_L] - [\psi_L^*][\sigma_a][\psi_R]\right)\right)^2$

$+ \left(-\frac{1}{2} Tr\left([\psi_R^*][\sigma_a][\psi_L] + [\psi_L^*][\sigma_a][\psi_R]\right)\right)^2$

$= \sum_{a=1}^3 \left|Tr\left([\psi_R^*][\sigma_a][\psi_L]\right)\right|^2 > 0$

4) The third term is $\sum_{b,\alpha\beta}\left(O^\beta_{p_b} O^\alpha_{q_b} - O^\beta_{q_b} O^\alpha_{p_b}\right)\left[\overrightarrow{\kappa}_a, G_\beta\right]^b O'^r_\alpha$

$\sum_{b,\alpha\beta}\left(O^\beta_{p_b} O^\alpha_{q_b} - O^\beta_{q_b} O^\alpha_{p_b}\right)\left[\overrightarrow{\kappa}_a, G_\beta\right]^b O'^r_\alpha$

$= \sum_{b,\beta}\left(\delta^r_{q_b} O^\beta_{p_b} - \delta^r_{p_b} O^\beta_{q_b}\right)\left[\overrightarrow{\kappa}_a, G_\beta\right]^b$

$= \sum_b \delta^r_{q_b}\left[\overrightarrow{\kappa}_a, G_{p_b}\right]^b - \delta^r_{p_b}\left[\overrightarrow{\kappa}_a, G_{q_b}\right]^b$

$= \sum_{bc=1}^6 G^b_{ac}\left(\delta^r_{q_b} G^c_{p_b} - \delta^r_{p_b} G^c_{q_b}\right)$

$\sum_{bc=1}^6 G^b_{ac}\left(\delta^r_{q_b} G^c_{p_b} - \delta^r_{p_b} G^c_{q_b}\right)$ can be computed explicitly :

a=1 : $G^b_{11}\left(\delta^r_{q_b} G^1_{p_b} - \delta^r_{p_b} G^1_{q_b}\right) + G^b_{12}\left(\delta^r_{q_b} G^2_{p_b} - \delta^r_{p_b} G^2_{q_b}\right) + G^b_{13}\left(\delta^r_{q_b} G^3_{p_b} - \delta^r_{p_b} G^3_{q_b}\right)$

$+G^b_{14}\left(\delta^r_{q_b} G^4_{p_b} - \delta^r_{p_b} G^4_{q_b}\right) + G^b_{15}\left(\delta^r_{q_b} G^5_{p_b} - \delta^r_{p_b} G^5_{q_b}\right) + G^b_{16}\left(\delta^r_{q_b} G^6_{p_b} - \delta^r_{p_b} G^6_{q_b}\right)$

a=2 :

$G^b_{21}\left(\delta^r_{q_b} G^1_{p_b} - \delta^r_{p_b} G^1_{q_b}\right) + G^b_{22}\left(\delta^r_{q_b} G^2_{p_b} - \delta^r_{p_b} G^2_{q_b}\right) + G^b_{23}\left(\delta^r_{q_b} G^3_{p_b} - \delta^r_{p_b} G^3_{q_b}\right)$

$+G^b_{24}\left(\delta^r_{q_b} G^4_{p_b} - \delta^r_{p_b} G^4_{q_b}\right) + G^b_{25}\left(\delta^r_{q_b} G^5_{p_b} - \delta^r_{p_b} G^5_{q_b}\right) + G^b_{26}\left(\delta^r_{q_b} G^6_{p_b} - \delta^r_{p_b} G^6_{q_b}\right)$

a=3 :

$G^b_{31}\left(\delta^r_{q_b} G^1_{p_b} - \delta^r_{p_b} G^1_{q_b}\right) + G^b_{32}\left(\delta^r_{q_b} G^2_{p_b} - \delta^r_{p_b} G^2_{q_b}\right) + G^b_{33}\left(\delta^r_{q_b} G^3_{p_b} - \delta^r_{p_b} G^3_{q_b}\right)$

$+G^b_{34}\left(\delta^r_{q_b} G^4_{p_b} - \delta^r_{p_b} G^4_{q_b}\right) + G^b_{35}\left(\delta^r_{q_b} G^5_{p_b} - \delta^r_{p_b} G^5_{q_b}\right) + G^b_{36}\left(\delta^r_{q_b} G^6_{p_b} - \delta^r_{p_b} G^6_{q_b}\right)$

a=4 :

$G^b_{41}\left(\delta^r_{q_b} G^1_{p_b} - \delta^r_{p_b} G^1_{q_b}\right) + G^b_{42}\left(\delta^r_{q_b} G^2_{p_b} - \delta^r_{p_b} G^2_{q_b}\right) + G^b_{43}\left(\delta^r_{q_b} G^3_{p_b} - \delta^r_{p_b} G^3_{q_b}\right)$

$+G^b_{44}\left(\delta^r_{q_b}G^4_{p_b}-\delta^r_{p_b}G^4_{q_b}\right)+G^b_{45}\left(\delta^r_{q_b}G^5_{p_b}-\delta^r_{p_b}G^5_{q_b}\right)+G^b_{46}\left(\delta^r_{q_b}G^6_{p_b}-\delta^r_{p_b}G^6_{q_b}\right)$
a=5 :
$G^b_{51}\left(\delta^r_{q_b}G^1_{p_b}-\delta^r_{p_b}G^1_{q_b}\right)+G^b_{52}\left(\delta^r_{q_b}G^2_{p_b}-\delta^r_{p_b}G^2_{q_b}\right)+G^b_{53}\left(\delta^r_{q_b}G^3_{p_b}-\delta^r_{p_b}G^3_{q_b}\right)$
$+G^b_{54}\left(\delta^r_{q_b}G^4_{p_b}-\delta^r_{p_b}G^4_{q_b}\right)+G^b_{55}\left(\delta^r_{q_b}G^5_{p_b}-\delta^r_{p_b}G^5_{q_b}\right)+G^b_{56}\left(\delta^r_{q_b}G^6_{p_b}-\delta^r_{p_b}G^6_{q_b}\right)$
a=6 :
$G^b_{61}\left(\delta^r_{q_b}G^1_{p_b}-\delta^r_{p_b}G^1_{q_b}\right)+G^b_{62}\left(\delta^r_{q_b}G^2_{p_b}-\delta^r_{p_b}G^2_{q_b}\right)+G^b_{63}\left(\delta^r_{q_b}G^3_{p_b}-\delta^r_{p_b}G^3_{q_b}\right)$
$+G^b_{64}\left(\delta^r_{q_b}G^4_{p_b}-\delta^r_{p_b}G^4_{q_b}\right)+G^b_{65}\left(\delta^r_{q_b}G^5_{p_b}-\delta^r_{p_b}G^5_{q_b}\right)+G^b_{66}\left(\delta^r_{q_b}G^6_{p_b}-\delta^r_{p_b}G^6_{q_b}\right)$
With the chosen basis in o(3,1) the structure coefficients are :
$G^b_{12}=\delta^b_3; G^b_{13}=-\delta^b_2; G^b_{14}=0; G^b_{15}=\delta^b_6; G^b_{16}=-\delta^b_5$
$G^b_{23}=\delta^b_1; G^b_{24}=-\delta^b_6; G^b_{25}=0; G^b_{26}=\delta^b_4$
$G^b_{34}=\delta^b_5; G^b_{35}=-\delta^b_4; G^b_{36}=0$
$G^b_{45}=-\delta^b_3; G^b_{46}=\delta^b_2$
$G^b_{56}=-\delta^b_1$
So :

a=1 : $\delta^b_3\left(\delta^r_{q_b}G^2_{p_b}-\delta^r_{p_b}G^2_{q_b}\right)-\delta^b_2\left(\delta^r_{q_b}G^3_{p_b}-\delta^r_{p_b}G^3_{q_b}\right)+\delta^b_6\left(\delta^r_{q_b}G^5_{p_b}-\delta^r_{p_b}G^5_{q_b}\right)-\delta^b_5\left(\delta^r_{q_b}G^6_{p_b}-\delta^r_{p_b}G^6_{q_b}\right)$

a=2 : $-\delta^b_3\left(\delta^r_{q_b}G^1_{p_b}-\delta^r_{p_b}G^1_{q_b}\right)+\delta^b_1\left(\delta^r_{q_b}G^3_{p_b}-\delta^r_{p_b}G^3_{q_b}\right)-\delta^b_6\left(\delta^r_{q_b}G^4_{p_b}-\delta^r_{p_b}G^4_{q_b}\right)+\delta^b_4\left(\delta^r_{q_b}G^6_{p_b}-\delta^r_{p_b}G^6_{q_b}\right)$

a=3 : $\delta^b_2\left(\delta^r_{q_b}G^1_{p_b}-\delta^r_{p_b}G^1_{q_b}\right)-\delta^b_1\left(\delta^r_{q_b}G^2_{p_b}-\delta^r_{p_b}G^2_{q_b}\right)+\delta^b_5\left(\delta^r_{q_b}G^4_{p_b}-\delta^r_{p_b}G^4_{q_b}\right)-\delta^b_4\left(\delta^r_{q_b}G^5_{p_b}-\delta^r_{p_b}G^5_{q_b}\right)$

a=4 : $\delta^b_6\left(\delta^r_{q_b}G^2_{p_b}-\delta^r_{p_b}G^2_{q_b}\right)-\delta^b_5\left(\delta^r_{q_b}G^3_{p_b}-\delta^r_{p_b}G^3_{q_b}\right)-\delta^b_3\left(\delta^r_{q_b}G^5_{p_b}-\delta^r_{p_b}G^5_{q_b}\right)+\delta^b_2\left(\delta^r_{q_b}G^6_{p_b}-\delta^r_{p_b}G^6_{q_b}\right)$

a=5 : $-\delta^b_6\left(\delta^r_{q_b}G^1_{p_b}-\delta^r_{p_b}G^1_{q_b}\right)+\delta^b_4\left(\delta^r_{q_b}G^3_{p_b}-\delta^r_{p_b}G^3_{q_b}\right)+\delta^b_3\left(\delta^r_{q_b}G^4_{p_b}-\delta^r_{p_b}G^4_{q_b}\right)-\delta^b_1\left(\delta^r_{q_b}G^6_{p_b}-\delta^r_{p_b}G^6_{q_b}\right)$

a=6 : $\delta^b_5\left(\delta^r_{q_b}G^1_{p_b}-\delta^r_{p_b}G^1_{q_b}\right)-\delta^b_4\left(\delta^r_{q_b}G^2_{p_b}-\delta^r_{p_b}G^2_{q_b}\right)-\delta^b_2\left(\delta^r_{q_b}G^4_{p_b}-\delta^r_{p_b}G^4_{q_b}\right)+\delta^b_1\left(\delta^r_{q_b}G^5_{p_b}-\delta^r_{p_b}G^5_{q_b}\right)$

One gets $\sum_{bc=1}^6 G^b_{ac}\left(\delta^r_{q_b}G^c_{p_b}-\delta^r_{p_b}G^c_{q_b}\right)=[T_G]^{ar}$ with the table T_G :

$$\begin{bmatrix} a\backslash r & 0 & 1 & 2 & 3 \\ 1 & (G^6_2-G^5_3) & (G^2_2+G^3_3) & (-G^2_1-G^6_0) & (G^5_0-G^3_1) \\ 2 & (G^4_3-G^6_1) & (G^6_0-G^1_2) & (G^1_1+G^3_3) & (-G^3_2-G^4_0) \\ 3 & (G^5_1-G^4_2) & (-G^5_0-G^1_3) & (G^4_0-G^2_3) & (G^2_2+G^1_1) \\ 4 & (G^3_2-G^2_3) & (-G^5_2-G^6_3) & (G^5_1-G^3_0) & (G^6_1+G^2_0) \\ 5 & (G^1_3-G^3_1) & (G^3_0+G^4_2) & (-G^4_1-G^6_3) & (G^6_2-G^1_0) \\ 6 & (G^2_1-G^1_2) & (G^4_3-G^2_0) & (G^1_0+G^5_3) & (-G^4_1-G^5_2) \end{bmatrix}$$

$$[T_G]^{ar} = \sum_{bc=1}^{6} G_{ac}^{b} \left(\delta_{q_b}^{r} G_{p_b}^{c} - \delta_{p_b}^{r} G_{q_b}^{c} \right) \tag{84}$$

5) The matter part of the Noether current Y_G is :
$N \sum_{a,r} \left(a_I \left([J] \left[\widetilde{\kappa}_a \right] \right)_r + V^r P_a \right) \partial_r \otimes \overrightarrow{\kappa}_a$ and :

$$Y_G = \sum_{a,r} \left(N \left(a_I \left([J] \left[\widetilde{\kappa}_a \right] \right)_r + a_D V^r P_a \right) + 2a_G \left[T_G \right]_r^a \right) \partial_r \otimes \overrightarrow{\kappa}_a \tag{85}$$

15.1.2 Superpotential

1) The superpotential $\varpi_4 (Z_G)$ is given by the equation 57 with
$Z_G = \sum_{a\{\alpha\beta\}} \frac{dL_F}{d\partial_\alpha G_\beta^a} \partial_\alpha \wedge \partial_\beta \otimes \overrightarrow{\kappa}_a = 2 \sum_{a\{\alpha\beta\}} \frac{dL_F}{d\mathcal{F}_{\alpha\beta}^a} \partial_\alpha \wedge \partial_\beta \otimes \overrightarrow{\kappa}_a$

It will be more convenient to isolate the constant a_G and denote from now on : $\varpi_4 (Z_G) = a_G \Pi_G$ so that :

$Z_G = 2 \sum_{a\{\alpha\beta\}} \left(O_{p_a}^{\beta} O_{q_a}^{\alpha} - O_{q_a}^{\beta} O_{p_a}^{\alpha} \right) \partial_\alpha \wedge \partial_\beta \otimes \overrightarrow{\kappa}_a$

$\Pi_G = \varpi_4 \left(2 \sum_{a\{\alpha\beta\}} \left(O_{p_a}^{\beta} O_{q_a}^{\alpha} - O_{q_a}^{\beta} O_{p_a}^{\alpha} \right) \partial_\alpha \wedge \partial_\beta \right) \otimes \overrightarrow{\kappa}_a$

2) Its exterior derivative is :

$d\Pi_G$

$= -4 \sum_{\alpha\beta=0}^{3} (-1)^{\alpha+1} \partial_\beta \left(\left(O_{p_a}^{\beta} O_{q_a}^{\alpha} - O_{q_a}^{\beta} O_{p_a}^{\alpha} \right) \det O' \right) dx^0..\wedge \widehat{dx^\alpha}..dx^3 \otimes \overrightarrow{\kappa}_a$

$\sum_{\beta=0}^{3} \partial_\beta \left(\left(O_{p_a}^{\beta} O_{q_a}^{\alpha} - O_{q_a}^{\beta} O_{p_a}^{\alpha} \right) \det O' \right)$

$= \sum_\beta \partial_\beta \left(\left(O_{p_a}^{\beta} O_{q_a}^{\alpha} - O_{q_a}^{\beta} O_{p_a}^{\alpha} \right) (\det O') \right) + \left(O_{p_a}^{\beta} O_{q_a}^{\alpha} - O_{q_a}^{\beta} O_{p_a}^{\alpha} \right) \partial_\beta \left((\det O') \right)$

$= (\det O') \sum_\beta O_{p_a}^{\beta} \partial_\beta O_{q_a}^{\alpha} - O_{q_a}^{\beta} \partial_\beta O_{p_a}^{\alpha} + O_{q_a}^{\alpha} \left(\partial_\beta O_{p_a}^{\beta} + O_{p_a}^{\beta} Tr \left(O \partial_\beta O' \right) \right) - O_{p_a}^{\alpha} \left(\partial_\beta O_{q_a}^{\beta} + O_{q_a}^{\beta} Tr \left(O \partial_\beta O' \right) \right)$

$= \left(\sum_{ra} O_r^{\alpha} c_a^r + \left(O_{q_a}^{\alpha} D_{p_a} - O_{p_a}^{\alpha} D_{q_a} \right) \right) (\det O')$ with $c_{p_a q_a}^r = c_a^r$

Where :

$\sum_{\alpha\beta} O_\alpha'^r \left(O_p^\beta \partial_\beta O_q^\alpha - O_q^\beta \partial_\beta O_p^\alpha \right) = [\partial_p, \partial_q]^r = c_{pq}^r$ are the structure coefficients of the basis ∂_i

$D_i = \sum_\beta \partial_\beta O_i^\beta + O_i^\beta Tr \left(O \partial_\beta O' \right) = \frac{1}{\det O'} \sum_\beta \partial_\beta \left(O_i^\beta (\det O') \right)$ is the divergence of the vector field ∂_i : $D_i \varpi_4 = \pounds_{\partial_i} \varpi_4 \Leftrightarrow D_i = Div (\partial_i)$

It is easy to check that : i=0,..3: $D_i = \sum_{j=0}^{3} c_{ji}^j$

$D_0 = c_{10}^1 + c_{20}^2 + c_{30}^3 = -c_4^1 - c_5^2 - c_6^3$

$D_1 = c^0_{01} + c^2_{21} + c^3_{31} = c^0_4 + c^2_3 - c^3_2$
$D_2 = c^0_{02} + c^1_{12} + c^3_{32} = c^0_5 - c^1_3 + c^3_1$
$D_0 = c^0_{03} + c^1_{13} + c^2_{23} = c^1_6 + c^1_2 - c^2_1$
So :
$d\Pi_G = -4\sum_{\alpha,r=0}^3 (-1)^{\alpha+1} \left(O^\alpha_r c^r_{p_a q_a} + O^\alpha_{q_a} D_{p_a} - O^\alpha_{p_a} D_{q_a}\right) \det O'$
$dx^0 \wedge . \wedge \widehat{dx^\alpha} .. \wedge dx^3 \otimes \overrightarrow{\kappa}_a$

$$d\Pi_G = -4\sum_{\alpha=0}^{3} (-1)^{\alpha+1} \sum_r \left(c^r_a + \delta^r_{q_a} D_{p_a} - \delta^r_{p_a} D_{q_a}\right) O^\alpha_r \det O' dx^0 \wedge .. \wedge \widehat{dx^\alpha} .. \wedge dx^3 \otimes \overrightarrow{\kappa}_a \tag{86}$$

3) Remark :$\varpi_4 (Z_G)$ can be computed directly with : $\varpi_4 = \partial^0 \wedge \partial^1 \wedge \partial^2 \wedge \partial^3$:

$\varpi_4 \left(\sum_a \partial_{p_a} \wedge \partial_{q_a} \otimes \overrightarrow{\kappa}_a\right)$
$= \left(\partial^1 \wedge \partial^2 \wedge \partial^3 \wedge \partial^4\right) \left(\sum_a \partial_{p_a} \wedge \partial_{q_a} \otimes \overrightarrow{\kappa}_a\right)$
$= \sum_a \left(\left(\partial^1 \wedge \partial^2 \wedge \partial^3 \wedge \partial^4\right) \left(\partial_{p_a} \wedge \partial_{q_a}\right)\right) \otimes \overrightarrow{\kappa}_a$
$\varpi_4 (Z_G) = -4a_G \varpi_4 \left(\sum_a \partial_{p_a} \wedge \partial_{q_a} \otimes \overrightarrow{\kappa}_a\right)$
$= 8a_G(\partial^0 \wedge \partial^1 \otimes \overrightarrow{\kappa}_1 + \partial^0 \wedge \partial^2 \otimes \overrightarrow{\kappa}_2 + \partial^0 \wedge \partial^3 \otimes \overrightarrow{\kappa}_3$
$+ \partial^3 \wedge \partial^2 \otimes \overrightarrow{\kappa}_4 + \partial^1 \wedge \partial^3 \otimes \overrightarrow{\kappa}_5 + \partial^2 \wedge \partial^1 \otimes \overrightarrow{\kappa}_6)$
$= 8a_G \left(\sum_{a=1}^3 \partial^{p_a} \wedge \partial^{q_a} \otimes \overrightarrow{\kappa}_{a+3} + \partial^{p_{a+3}} \wedge \partial^{q_{a+3}} \otimes \overrightarrow{\kappa}_a\right)$

This quantity can be expressed with the Hodge dual of : $\sum_{a=1}^6 \eta_{p_a p_a} \partial^{p_a} \wedge \partial^{q_a} \otimes \overrightarrow{\kappa}_a$:

$\varpi_4 (Z_G) = 8a_G * (-\partial^0 \wedge \partial^1 \otimes \overrightarrow{\kappa}_1 - \partial^0 \wedge \partial^2 \otimes \overrightarrow{\kappa}_2 - \partial^0 \wedge \partial^3 \otimes \overrightarrow{\kappa}_3$
$+ \partial^3 \wedge \partial^2 \otimes \overrightarrow{\kappa}_4 + \partial^1 \wedge \partial^3 \otimes \overrightarrow{\kappa}_5 + \partial^2 \wedge \partial^1 \otimes \overrightarrow{\kappa}_6)$
$\varpi_4 (Z_G) = 8a_G * \left(\sum_{a=1}^6 \eta_{p_a p_a} \partial^{p_a} \wedge \partial^{q_a} \otimes \overrightarrow{\kappa}_a\right)$

15.1.3 Equation

This equation is linear in G^a_r and thus can be solved explicitly.

1) The gravitational equation reads :
$\varpi_4 (Y_G) = \frac{1}{2} a_G d\Pi_G$ or $\forall a, \alpha : Y^{a\alpha}_G = -2\sum_{\beta b} \frac{1}{\det O'} \partial_\beta \left(\frac{dL_F(\det O')}{d\mathcal{F}^a_{G\alpha\beta}}\right)$
with $Y^{ar}_G = \sum_{a,r} \left(N \left(a_I \left([J]\left[\widetilde{\kappa}_a\right]\right)_r + a_D V^r P_a\right) + 2a_G \left[T_G\right]^{ar}\right)$
$\left(N \left(a_I \left([J]\left[\widetilde{\kappa}_a\right]\right)_r + a_D V^r P_a\right) + 2a_G \left[T_G\right]^{ar}\right) O^\alpha_r$
$= -2a_G \sum_r \left(c^r_{p_a q_a} + \delta^r_{q_a} D_{p_a} - \delta^r_{p_a} D_{q_a}\right) O^\alpha_r$

That is :

$$[T_G]^{ar} = -c_a^r + \delta_{p_a}^r D_{q_a} - \delta_{q_a}^r D_{p_a} - \frac{N}{2a_G}\left(a_I\left([J]\left[\widetilde{\kappa}_a\right]\right)_r + a_D V^r P_a\right) \tag{87}$$

where $[T_G]$ is the previous table. We have 24 linear equations which are with

$$K_a^r = \frac{N}{2a_G}\left(a_I\left([J]\left[\widetilde{\kappa}_a\right]\right)_r + a_D V^r P_a\right) \tag{88}$$

$\sum_{bc=1}^{6} G_{ac}^{b}\left(\delta_{q_b}^r G_{p_b}^c - \delta_{p_b}^r G_{q_b}^c\right) = -c_a^r + \delta_{p_a}^r D_{q_a} - \delta_{q_a}^r D_{p_a} - K_a^r$

$$\begin{bmatrix} a & r & LHS & RHS \\ 1 & 1 & G_2^2 + G_3^3 & -c_1^1 - K_1^1 \\ 1 & 2 & -G_1^2 - G_0^6 & -D_3 - c_1^2 - K_1^2 \\ 1 & 3 & -G_1^3 + G_0^5 & D_2 - c_1^3 - K_1^3 \\ 1 & 0 & G_2^6 - G_3^5 & -c_1^0 - K_1^0 \\ 2 & 1 & -G_2^1 + G_0^6 & -c_2^1 + D_3 - K_2^1 \\ 2 & 2 & G_1^1 + G_3^3 & -c_2^2 - K_2^2 \\ 2 & 3 & -G_2^3 - G_0^4 & -D_1 - c_2^3 - K_2^3 \\ 2 & 0 & -G_1^6 + G_3^4 & -c_2^0 - K_2^0 \\ 3 & 1 & -G_3^1 - G_0^5 & -c_3^1 - D_2 - K_3^1 \\ 3 & 2 & -G_3^2 + G_0^4 & D_1 - c_3^2 - K_3^2 \\ 3 & 3 & G_1^1 + G_2^2 & -c_3^3 - K_3^3 \\ 3 & 0 & G_1^5 - G_2^4 & -c_3^0 - K_3^0 \\ 4 & 1 & -G_2^5 - G_3^6 & -c_4^1 - D_0 - K_4^1 \\ 4 & 2 & G_1^5 - G_0^3 & -c_4^2 - K_4^2 \\ 4 & 3 & G_4^2 + G_1^6 & -c_4^3 - K_4^3 \\ 4 & 0 & G_2^3 - G_3^2 & D_1 - c_4^0 - K_4^0 \\ 5 & 1 & G_2^4 + G_4^3 & -c_5^1 - K_5^1 \\ 5 & 2 & -G_1^4 - G_3^6 & -D_0 - c_5^2 - K_5^2 \\ 5 & 3 & -G_4^1 + G_2^6 & -c_5^3 - K_5^3 \\ 5 & 0 & G_3^1 - G_1^3 & D_2 - c_5^0 - K_5^0 \\ 6 & 1 & -G_4^2 + G_3^4 & -c_6^1 - K_6^1 \\ 6 & 2 & G_4^1 + G_3^5 & -c_6^2 - K_6^2 \\ 6 & 3 & -G_1^4 - G_2^5 & -D_0 - c_6^3 - K_6^3 \\ 6 & 0 & -G_2^1 + G_1^2 & D_3 - c_6^0 - K_6^0 \end{bmatrix}$$

and the solution is : $2G_r^a =$

$$\begin{bmatrix} 2G_r^a & r=0 & & r=1 \\ G_0^1 & -K_1^0+K_5^3-K_6^2-c_1^0+c_5^3-c_6^2 & G_1^1 & K_1^1+c_1^1-K_2^2-K_3^3-c_2^2-c_3^3 \\ G_0^2 & K_6^1+c_6^1-K_2^0-K_4^3-c_2^0-c_4^3 & G_1^2 & K_2^1+2c_2^1+K_1^2-K_6^0 \\ G_0^3 & -K_5^1-c_5^1+K_4^2-K_3^0+c_4^2-c_3^0 & G_1^3 & K_3^1+2c_3^1+K_1^3+K_5^0 \\ G_0^4 & K_2^3-K_3^2+K_4^0+2c_4^0 & G_1^4 & -K_4^1-2c_4^1+K_5^2+K_6^3 \\ G_0^5 & K_3-K_1^3+K_5^0+2c_5^0 & G_1^5 & -K_5^1-c_5^1-K_4^2-K_3^0-c_4^2-c_3^0 \\ G_0^6 & -K_2+K_1^2+K_6^0+2c_6^0 & G_1^6 & -K_6^1-c_6^1+K_2^0-K_4^3+c_2^0-c_4^3 \end{bmatrix}$$

$$\begin{bmatrix} & r=2 & & r=3 \\ G_2^1 & K_2^1+K_1^2+K_6^0+2c_1^2 & G_3^1 & K_3+K_1^3-K_5^0+2c_1^3 \\ G_2^2 & -K_1^1-c_1^1+K_2^2-K_3^3+c_2^2-c_3^3 & G_3^2 & K_2^3+K_3^2+K_4^0+2c_2^3 \\ G_2^3 & K_2^3+K_3^2-K_4^0+2c_3^2 & G_3^3 & -K_1^1-c_1^1-K_2^2+K_3^3-c_2^2+c_3^3 \\ G_2^4 & -K_5^1-c_5^1-K_4^2+K_3^0-c_4^2+c_3^0 & G_3^4 & -K_6^1-c_6^1-K_2^0-K_4^3-c_2^0-c_4^3 \\ G_2^5 & K_4^1-K_5^2+K_6^3-2c_5^2 & G_3^5 & K_1^0-K_5^3-K_6^2+c_1^0-c_5^3-c_6^2 \\ G_2^6 & -K_1^0-K_5^3-K_6^2-c_1^0-c_5^3-c_6^2 & G_3^6 & K_4^1+K_5^2-K_6^3-2c_6^3 \end{bmatrix}$$

Notice that in this equation ony the momentum K appears, and not ψ per se.

In the vacuum the gravitation field is not null (in accordance with the General Relativity) and entirely given by the structure coefficients which thus fully represent the geometry of the universe.

2) The Noether current : $Y_G^{ar} = -2a_G \sum_r \left(c_{p_a q_a}^r + \delta_{q_a}^r D_{p_a} - \delta_{p_a}^r D_{q_a} \right)$ is expressed with the structure coefficients only.

We have the table :

$$Y_G^{ar} = 2a_G \begin{bmatrix} a\backslash r & 0 & 1 & 2 & 3 \\ 1 & -c_1^0 & -c_1^1 & -\left(c_2^1+c_6^0\right) & \left(-c_3^1+c_5^0\right) \\ 2 & -c_2^0 & \left(-c_1^2+c_6^0\right) & -c_2^2 & -\left(c_3^2+c_4^0\right) \\ 3 & -c_3^0 & -\left(c_1^3+c_5^0\right) & \left(-c_2^3+c_4^0\right) & -c_3^3 \\ 4 & \left(c_3^2-c_2^3\right) & \left(c_5^2+c_6^3\right) & -c_4^2 & -c_4^3 \\ 5 & \left(-c_3^1+c_1^3\right) & -c_5^1 & \left(c_4^1+c_6^3\right) & -c_5^3 \\ 6 & \left(c_2^1-c_1^2\right) & -c_6^1 & -c_6^2 & \left(c_4^1+c_5^2\right) \end{bmatrix}$$

The flux of this vector through the S(t) hypersurfaces is conserved. That is $\int_{S(0)} \varpi_4 \left(Y_G\right) = \int_{S(t)} \varpi_4 \left(Y_G\right)$

15.1.4 Symmetry

The value of the torsion tensor $\nabla_e \Theta = \sum_{r,a} \left(2c^r_{p_a q_a} + T^{ar}\right) \partial^{p_a} \wedge \partial^{q_a} \otimes \partial_r$ has been calculated previously (table 2).

Using the results above we get the table:

$$(\nabla_e \Theta)^r_{p_a q_a} = 3\left[c^r_a\right] + [K]^r_a + \frac{1}{2}\begin{bmatrix} 0 & 0 & \Theta_3 & -\Theta_2 \\ 0 & -\Theta_3 & 0 & \Theta_1 \\ 0 & \Theta_2 & -\Theta_1 & 0 \\ -\Theta_1 & \Theta_0 & 0 & 0 \\ -\Theta_2 & 0 & \Theta_0 & 0 \\ -\Theta_3 & 0 & 0 & \Theta_0 \end{bmatrix}$$

with : $\Theta_0 = -\left(K^1_4 + K^2_5 + K^3_6\right); \Theta_1 = -K^3_2 + K^2_3 + K^0_4; \Theta_2 = -K^1_3 + K^3_1 + K^0_5; \Theta_3 = -K^2_1 + K^1_2 + K^0_6$

$\Theta_0 = -\sum_{a=1}^3 K^a_{a+3}; k = 1,2,3 : \Theta_k = K^{q_k}_{p_k} - K^{p_k}_{q_k} + K^0_{k+3}$

So in the vacuum the torsion is given by the structure coefficients. And the connection is torsion free iff the particles have some specific distribution.

15.2 Other force fields

We will define two moments : the "charge" ρ_a and the "magnetic moment" $\overrightarrow{\mu}_a$, both computed from the state tensor, gauge and chart invariant. The law for the fields take then a simple, geometric form, independant from the gravitational field:

$N\varpi_4 \left(a_D \rho_a \overrightarrow{V} - i a_I \overrightarrow{\mu}_a\right) = \nabla_e * \overline{\mathcal{F}}^a_A$

15.2.1 Noether currents

1) The Noether currents are : $Y^a_{AR} = \sum_\alpha Y^{\alpha a}_{AR} \partial_\alpha, Y^a_{AI} = \sum_\alpha Y^{\alpha a}_{AI} \partial_\alpha$ with :

$Y^{\alpha a}_{AR} = V \sum_{ij} \frac{dL_M}{d \operatorname{Re} \nabla_\alpha \psi^{ij}} \operatorname{Re}\left(\left[\psi^\diamond\right]\left[\theta_a\right]^t\right)^i_j + \frac{dL_M}{d \operatorname{Im} \nabla_\alpha \psi^{ij}} \operatorname{Im}\left(\left[\psi^\diamond\right]\left[\theta_a\right]^t\right)^i_j$

$+2\sum_{b\beta} \frac{dL_F}{d \operatorname{Re} \mathcal{F}^b_{A,\alpha\beta}} \operatorname{Re}\left[\overrightarrow{\theta}_a, \grave{A}_\beta\right]^b + \frac{dL_F}{d \operatorname{Im} \mathcal{F}^b_{A,\alpha\beta}} \operatorname{Im}\left[\overrightarrow{\theta}_a, \grave{A}_\beta\right]^b$

$Y^{\alpha a}_{AI} = V \sum_{ij} -\frac{dL_M}{d \operatorname{Re} \nabla_\alpha \psi^{ij}} \operatorname{Im}\left(\left[\psi^\diamond\right]\left[\theta_a\right]^t\right)^i_j + \frac{dL_M}{d \operatorname{Im} \nabla_\alpha \psi^{ij}} \operatorname{Re}\left(\left[\psi^\diamond\right]\left[\theta_a\right]^t\right)^i_j$

$+2\sum_{b\beta} -\frac{dL_F}{d \operatorname{Re} \mathcal{F}^b_{A,\alpha\beta}} \operatorname{Im}\left[\overrightarrow{\theta}_a, \grave{A}_\beta\right]^b + \frac{dL_F}{d \operatorname{Im} \mathcal{F}^b_{A,\alpha\beta}} \operatorname{Re}\left[\overrightarrow{\theta}_a, \grave{A}_\beta\right]^b$

We have already :

$\frac{dL_M}{d \operatorname{Re} \nabla_\alpha \psi^{ij}} = N \operatorname{Im}\left([\psi]^* [\gamma_0] [D^\alpha_M]\right)^j_i$

$\frac{dL_M}{d\operatorname{Im}\nabla_\alpha\psi^{ij}} = N\operatorname{Re}\left([\psi]^*[\gamma_0][D^\alpha_M]\right)^j_i$

Computation of the derivatives with respect to $\operatorname{Re}\mathcal{F}^b_{A,\alpha\beta}, \operatorname{Im}\mathcal{F}^b_{A,\alpha\beta}$

$L_F = a_F\frac{1}{2}\sum_a\sum_{\eta\xi\lambda\mu} g^{\eta\lambda}g^{\xi\mu}\overline{\mathcal{F}}^a_{A\eta\xi}\mathcal{F}^a_{A\lambda\mu} + a_G R$

$= a_F\frac{1}{2}\sum_a\sum_{\eta\xi\lambda\mu} g^{\eta\lambda}g^{\xi\mu}\left(\operatorname{Re}\mathcal{F}^a_{A\eta\xi} - i\operatorname{Im}\mathcal{F}^a_{A\eta\xi}\right)\left(\operatorname{Re}\mathcal{F}^a_{A\lambda\mu} + i\operatorname{Im}\mathcal{F}^a_{A\lambda\mu}\right) + a_G R$

$\frac{dL_F}{d\operatorname{Re}\mathcal{F}^b_{A,\alpha\beta}} = \frac{1}{2}a_F\sum_{\eta\xi\lambda\mu} g^{\eta\lambda}g^{\xi\mu}\left(\delta^\alpha_\eta\delta^\beta_\xi\overline{\mathcal{F}}^b_{A\lambda\mu} + \mathcal{F}^b_{A\eta\xi}\delta^\alpha_\lambda\delta^\beta_\mu\right) = \frac{1}{2}a_F\left(\overline{\mathcal{F}}^{b\alpha\beta}_A + \mathcal{F}^{b,\alpha\beta}_A\right)$

$\frac{dL_F}{d\operatorname{Re}\mathcal{F}^b_{A,\alpha\beta}} = a_F\operatorname{Re}\mathcal{F}^{b,\alpha\beta}_A$

$\frac{dL_F}{d\operatorname{Im}\mathcal{F}^b_{A,\alpha\beta}} = \frac{1}{2}a_F\sum_{\eta\xi\lambda\mu} g^{\eta\lambda}g^{\xi\mu}\left(-i\delta^\alpha_\eta\delta^\beta_\xi\mathcal{F}^b_{A\lambda\mu} + i\overline{\mathcal{F}}^a_{A\eta\xi}\delta^\alpha_\lambda\delta^\beta_\mu\right)$

$= \frac{1}{2}ia_F\sum_{\eta\xi\lambda\mu}\left(-g^{\alpha\lambda}g^{\beta\mu}\mathcal{F}^b_{A\lambda\mu} + g^{\eta\alpha}g^{\xi\beta}\overline{\mathcal{F}}^a_{A\eta\xi}\right)$

$= i\frac{1}{2}a_F\left(-\mathcal{F}^{b\alpha\beta}_A + \overline{\mathcal{F}}^{b,\alpha\beta}_A\right) = i\frac{1}{2}a_F\left(-2i\operatorname{Im}\mathcal{F}^{b,\alpha\beta}_A\right)$

$\frac{dL_F}{d\operatorname{Im}\mathcal{F}^b_{A,\alpha\beta}} = a_F\operatorname{Im}\mathcal{F}^{b,\alpha\beta}_A$

So the equations are :

$Y^{\alpha a}_{AR} = N\sum_{ij}\operatorname{Im}\left([\psi]^*[\gamma_0][D^\alpha_M]\right)^j_i\operatorname{Re}\left([\psi][\theta_a]^t\right)^i_j + \operatorname{Re}\left([\psi]^*[\gamma_0][D^\alpha_M]\right)^j_i\operatorname{Im}\left(\left[\psi^\Diamond\right][\theta_a]^t\right)^i_j$

$+2a_F\sum_{b\beta}\operatorname{Re}\mathcal{F}^{b,\alpha\beta}_A\operatorname{Re}\left[\overrightarrow{\theta}_a, \grave{A}_\beta\right]^b + \operatorname{Im}\mathcal{F}^{b,\alpha\beta}_A\operatorname{Im}\left[\overrightarrow{\theta}_a, \grave{A}_\beta\right]^b$

$= N\operatorname{Im}Tr[\psi]^*[\gamma_0][D^\alpha_M][\psi][\theta_a]^t + 2a_F\sum_{b\beta}\operatorname{Re}\left(\mathcal{F}^{b,\alpha\beta}_A\overline{\left[\overrightarrow{\theta}_a, \grave{A}_\beta\right]^b}\right)$

$Y^{\alpha a}_{AR} = N\operatorname{Im}Tr[\psi]^*[\gamma_0][D^\alpha_M][\psi][\theta_a]^t + 2a_F\sum_{b\beta}\operatorname{Re}\left(\left[\overrightarrow{\theta}_a, \grave{A}_\beta\right], \mathcal{F}^{\alpha\beta}_A\right)$

$Y^{\alpha a}_{AI} = N\sum_{ij} -\operatorname{Im}\left([\psi]^*[\gamma_0][D^\alpha_M]\right)^j_i\operatorname{Im}\left([\psi][\theta_a]^t\right)^i_j + \operatorname{Re}\left([\psi]^*[\gamma_0][D^\alpha_M]\right)^j_i\operatorname{Re}\left([\psi][\theta_a]^t\right)^i_j$

$+2a_F\sum_{b\beta} -\operatorname{Re}\mathcal{F}^{b,\alpha\beta}_A\operatorname{Im}\left[\overrightarrow{\theta}_a, \grave{A}_\beta\right]^b + \operatorname{Im}\mathcal{F}^{b,\alpha\beta}_A\operatorname{Re}\left[\overrightarrow{\theta}_a, \grave{A}_\beta\right]^b$

$= N\operatorname{Re}Tr[\psi]^*[\gamma_0][D^\alpha_M][\psi][\theta_a]^t + 2a_F\sum_{b\beta}\operatorname{Im}\left(\mathcal{F}^{b,\alpha\beta}_A\overline{\left[\overrightarrow{\theta}_a, \grave{A}_\beta\right]^b}\right)$

$Y^{\alpha a}_{AR} = N\operatorname{Re}Tr[\psi]^*[\gamma_0][D^\alpha_M][\psi][\theta_a]^t + 2a_F\sum_{b\beta}\operatorname{Im}\left(\left[\overrightarrow{\theta}_a, \grave{A}_\beta\right], \mathcal{F}^{\alpha\beta}_A\right)$

$\sum_b\mathcal{F}^{b,\alpha\beta}_A\overline{\left[\overrightarrow{\theta}_a, \grave{A}_\beta\right]^b}$ is the scalar product $\left(\left[\overrightarrow{\theta}_a, \grave{A}_\beta\right], \mathcal{F}^{\alpha\beta}_A\right)$ in T_1U^c and with our assumptions that the scalar product is preserved by the adjoint operator : $\forall\overrightarrow{\theta}, \overrightarrow{\theta}_1, \overrightarrow{\theta}_2 \in T_1U^c : \left(\left[\overrightarrow{\theta}, \overrightarrow{\theta}_1\right], \overrightarrow{\theta}_2\right) = -\left(\overrightarrow{\theta}_1, \left[\overrightarrow{\theta}, \overrightarrow{\theta}_2\right]\right)$

$\left(\left[\overrightarrow{\theta}_a, \grave{A}_\beta\right], \mathcal{F}^{\alpha\beta}_A\right) = -\left(\left[\grave{A}_\beta, \overrightarrow{\theta}_a\right], \mathcal{F}^{\alpha\beta}_A\right) = \left(\overrightarrow{\theta}_a, \left[\grave{A}_\beta, \mathcal{F}^{\alpha\beta}_A\right]\right) = \left[\grave{A}_\beta, \mathcal{F}^{\alpha\beta}_A\right]^a$

Thus :

$Y^{\alpha a}_{AR} = N\operatorname{Im}Tr[\psi]^*[\gamma_0][D^\alpha_M][\psi][\theta_a]^t + 2a_F\sum_\beta\operatorname{Re}\left[\grave{A}_\beta, \mathcal{F}^{\alpha\beta}_A\right]^a$

$Y^{\alpha a}_{AI} = N\operatorname{Re}Tr[\psi]^*[\gamma_0][D^\alpha_M][\psi][\theta_a]^t + 2a_F\sum_\beta\operatorname{Im}\left[\grave{A}_\beta, \mathcal{F}^{\alpha\beta}_A\right]^a$

2) Let us compute the first terms : $\operatorname{Im} Tr\,[\psi]^*[D_M^\alpha][\psi][\theta_a]^t, \operatorname{Re} Tr\,[\psi]^*[D_M^\alpha][\psi][\theta_a]^t$

$Tr\,[\psi]^*[D_M^\alpha][\psi][\theta_a]^t$

$= a_I \sum_r O_r^\alpha \operatorname{Im} Tr\left([\psi]^*[\gamma_0\gamma^r][\psi][\theta_a]^t\right) + a_D V^\alpha \operatorname{Im} Tr\left([\psi^*][\gamma_0][\psi][\theta_a]^t\right)$

a) $[\psi]^*[\gamma_0\gamma^r][\psi][\theta_a]^t = i\left(\eta^{rr}[\psi_R]^*\sigma_r[\psi_R][\theta_a]^t - [\psi_L]^*\sigma_r[\psi_L][\theta_a]^t\right)$

So : $Tr\left([\psi]^*[\gamma_0\gamma^r][\psi][\theta_a]^t\right) = iTr\left(\left(\eta^{rr}[\psi_R]^*\sigma_r[\psi_R][\theta_a]^t - [\psi_L]^*\sigma_r[\psi_L][\theta_a]^t\right)\right)$

We have assumed that the representation (W,χ) is unitary so $[\theta_a]$ is antihermitian :$[\theta_a]^* = -[\theta_a], [\theta_a]^t = -\overline{[\theta_a]}$ and :

$\overline{Tr\left([\psi_R]^*\sigma_r[\psi_R][\theta_a]^t\right)} = Tr\left([\psi_R]^*\sigma_r[\psi_R][\theta_a]^t\right)^* = -Tr\left([\psi_R]^t\sigma_r^t\overline{[\psi_R]}[\theta_a]\right)$

$= -Tr\left([\theta_a]^t[\psi_R]^*\sigma_r[\psi_R]\right) = -Tr\left([\psi_R]^*\sigma_r[\psi_R][\theta_a]^t\right)$

Each quantity $Tr\left([\psi_R]^*\sigma_r[\psi_R][\theta_a]^t\right), Tr\left([\psi_L]^*\sigma_r[\psi_L][\theta_a]^t\right)$ is an imaginary scalar.

Let be :

$\left\langle\psi,[\gamma_0\gamma^r][\psi][\theta_a]^t\right\rangle = iTr\left(\eta^{rr}[\psi_R]^*\sigma_r[\psi_R][\theta_a]^t - [\psi_L]^*\sigma_r[\psi_L][\theta_a]^t\right) = -[\mu]_a^r$

$[\mu]_a^r = -iTr\left(\eta^{rr}[\psi_R]^*\sigma_r[\psi_R][\theta_a]^t\right) + iTr\left([\psi_L]^*\sigma_r[\psi_L][\theta_a]^t\right)$

$$\left\langle\psi,[\gamma_0\gamma^r][\psi][\theta_a]^t\right\rangle = \mathbf{iTr}\left(\eta^{rr}[\psi_R]^*\sigma_r[\psi_R][\theta_a]^t - [\psi_L]^*\sigma_r[\psi_L][\theta_a]^t\right) = -[\mu]_a^r \tag{89}$$

b) $[\psi^*][\gamma_0][\psi][\theta_a]^t = i\left([\psi_R]^*[\psi_L][\theta_a]^t - [\psi_L]^*[\psi_R][\theta_a]^t\right)$

So : $Tr\left([\psi^*][\gamma_0][\psi][\theta_a]^t\right) = iTr\left([\psi_R]^*[\psi_L][\theta_a]^t - [\psi_L]^*[\psi_R][\theta_a]^t\right)$

With the same assumption about $[\theta_a]$ as above :

$\overline{Tr\,[\psi_R]^*[\psi_L][\theta_a]^t} = Tr\left(\left([\psi_R]^*[\psi_L][\theta_a]^t\right)^*\right)$

$= Tr\left(\overline{[\theta_a]}[\psi_L]^*[\psi_R]\right) = -Tr\left([\theta_a]^t[\psi_L]^*[\psi_R]\right) = -Tr\left([\psi_L]^*[\psi_R][\theta_a]^t\right)$

So one cannot tell much about the individual quantities

$Tr\left([\psi_L]^*[\psi_R][\theta_a]^t\right), Tr\left([\psi_R]^*[\psi_L][\theta_a]^t\right)$

but that they are complex conjugates of each other. So the difference is a real number:

$\overline{Tr\left([\psi_R]^*[\psi_L][\theta_a]^t - [\psi_L]^*[\psi_R][\theta_a]^t\right)}$

$= \overline{Tr\left([\psi_R]^*[\psi_L][\theta_a]^t\right)} - \overline{Tr\left([\psi_L]^*[\psi_R][\theta_a]^t\right)}$

$= -Tr\left([\psi_L]^*[\psi_R][\theta_a]^t + [\psi_R]^*[\psi_L][\theta_a]^t\right)$

$= -Tr\left([\psi_L]^*[\psi_R][\theta_a]^t - [\psi_R]^*[\psi_L][\theta_a]^t\right)$

$= Tr\left([\psi_R]^*[\psi_L][\theta_a]^t - [\psi_L]^*[\psi_R][\theta_a]^t\right) \in \mathbb{R}$

So let be :

$$\left\langle \psi, [\gamma_0\gamma^r] [\psi] [\theta_a]^t \right\rangle = \mathbf{i}\rho_a = \mathbf{iTr} \left([\psi_R]^* [\psi_L] [\theta_a]^t - [\psi_L]^* [\psi_R] [\theta_a]^t \right) \quad (90)$$

3) The currents read :

$Y_{AR}^{\alpha a} = N a_D V^\alpha \rho_a + 2a_F \sum_\beta \mathrm{Re} \left[\grave{A}_\beta, \mathcal{F}_A^{\alpha\beta} \right]^a$

$Y_{AI}^{\alpha a} = -N a_I \sum_r O_r^\alpha [\mu_R - \mu_L]_a^r + 2a_F \sum_\beta \mathrm{Im} \left[\grave{A}_\beta, \mathcal{F}_A^{\alpha\beta} \right]^a$

We can combine both currents $Y_A^a = \sum_\alpha \left(Y_{AR}^{a\alpha} + i Y_{AI}^{a\alpha} \right) \partial_\alpha$ and denote :

$$Y_A^a = N \left(a_D \sum_\alpha V^\alpha \rho_a \partial_\alpha - i a_I \sum_r [\mu_A]_a^r \partial_r \right) + 2a_F \sum_{\alpha\beta} \left[\grave{A}_\beta, \mathcal{F}_A^{\alpha\beta} \right]^a \partial_\alpha \quad (91)$$

ρ_a is similar to a "charge" of the particle, and μ_A to a "magnetic moment". The a index is related to the kind of force. These quantities are functions on M and therefore invariant by a change of gauge. The matter part of the Noether current is :

$N \sum_\alpha \left(a_D V^\alpha \rho_a - i a_I \sum_r O_r^\alpha [\mu_A]_a^r \right) \partial_\alpha$

The Noether current is a geometric quantity. As we see in the formula above the pertinent geometric quantities, pertaining to the particle, are the 4-vectors :

- the "charge current" linked with the velocity : $\rho_a \overrightarrow{V} = \sum_\alpha V^\alpha \rho_a \partial_\alpha$

- the "magnetic moment" linked with the tetrad : $\overrightarrow{\mu}_a = \sum_r [\mu_A]_a^r \partial_r$

Remark : these equations do no involve gravitation. They can be seen as relating the fields to the sources. The gravitational field comes back through the trajectories of the particles.

15.2.2 Superpotential

Implementing the previous definitions :

$\Pi_{AR}^a = 4a_F (\det O') \sum_{\lambda<\mu} \sum_{\alpha<\beta,a} \epsilon(\lambda,\mu,\alpha,\beta) \mathrm{Re}\, \mathcal{F}_A^{a,\alpha\beta} dx^\lambda \wedge dx^\mu$

$\Pi_{AI}^a = 4a_F (\det O') \sum_{\lambda<\mu} \sum_{\alpha<\beta,a} \epsilon(\lambda,\mu,\alpha,\beta) \mathrm{Im}\, \mathcal{F}_A^{a,\alpha\beta} dx^\lambda \wedge dx^\mu$

Let us introduce :

$\Pi_A^a = \Pi_{AR}^a + i\Pi_{AI}^a = 4 (\det O') a_F \sum_{\lambda<\mu} \sum_{\alpha<\beta,a} \epsilon(\lambda,\mu,\alpha,\beta) \mathcal{F}_A^{a\alpha\beta} dx^\lambda \wedge dx^\mu$

$\Pi_A^a = -4 (\det O') a_F \{ \mathcal{F}^{a,01} dx^3 \wedge dx^2 + \mathcal{F}^{a,02} dx^1 \wedge dx^3 + \mathcal{F}^{a,03} dx^2 \wedge dx^1$
$+ \mathcal{F}^{a,32} dx^0 \wedge dx^1 + \mathcal{F}^{a,13} dx^0 \wedge dx^2 + \mathcal{F}^{a,21} dx^0 \wedge dx^3 \}$

We can recognize the Hodge dual of the conjugate of the curvature form (cf 74):

So we can write :

$$\Pi_A^a = 4a_F\left(*\overline{\mathcal{F}}_A^a\right) \tag{92}$$

15.2.3 Equations

1) The equations 63 take here the simple form :

$\varpi_4\left(Y_{AR}^a\right) = \frac{1}{2}d\Pi_{AR}^a = \operatorname{Re}\varpi_4\left(Y_A^a\right) = \frac{1}{2}d\operatorname{Re}\Pi_{AR}^a; \varpi_4\left(Y_{AI}^a\right) = \frac{1}{2}d\Pi_{AI}^a = \operatorname{Im}\varpi_4\left(Y_{AI}^a\right) = \frac{1}{2}d\operatorname{Im}\Pi_{AI}^a$

$$\varpi_4\left(Y_A^a\right) = 2a_F d\left(*\overline{\mathcal{F}}_A^a\right) \tag{93}$$

2) The equations read :

$\forall a,\alpha : Y_{AR}^{\alpha a} = -2\frac{1}{\det O'}\sum_\beta \partial_\beta\left(\frac{dL_F(\det O')}{d\operatorname{Re}\mathcal{F}_{A,\alpha\beta}^a}\right) = -2\frac{1}{\det O'}a_F\sum_\beta \operatorname{Re}\partial_\beta\left(\mathcal{F}_A^{a\alpha\beta}\left(\det O'\right)\right)$

$\forall a,\alpha : Y_{AI}^{\alpha a} = -2\frac{1}{\det O'}\sum_\beta \partial_\beta\left(\frac{\partial L_F(\det O')}{\partial\operatorname{Im}\mathcal{F}_{A,\alpha\beta}^a}\right) = -2\frac{1}{\det O'}a_F\sum_\beta \operatorname{Im}\partial_\beta\left(\mathcal{F}_A^{a\alpha\beta}\left(\det O'\right)\right)$

That we can write :

$\forall a,\alpha : Y_A^{\alpha a} = -2\frac{1}{\det O'}a_F\sum_\beta \partial_\beta\left(\mathcal{F}_A^{a\alpha\beta}\left(\det O'\right)\right)$

Or :

$$\begin{gathered}\forall a,\alpha : N\left(a_D V^\alpha\rho_a - ia_I\sum_r O_r^\alpha\left[\mu_A\right]_a^r\right)\det O' \\ = -2a_F\sum_\beta\left(\left[\grave{A}_\beta,\mathcal{F}_A^{\alpha\beta}\det O'\right]^a + \partial_\beta\left(\mathcal{F}_A^{a\alpha\beta}\left(\det O'\right)\right)\right)\end{gathered} \tag{94}$$

3) This equation can also be written as :

$\varpi_4\left(Y_A^a\right) = 2a_F d\left(*\mathcal{F}_A^a\right)$

$= N\varpi_4\left(a_D\sum_\alpha V^\alpha\rho_a\partial_\alpha - ia_I\sum_r\left[\mu_R-\mu_L\right]_a^r\partial_r\right) + 2a_F\varpi_4\left(\sum_\beta\left[\grave{A}_\beta,\mathcal{F}_A^{\alpha\beta}\right]^a\partial_\alpha\right)$

$N\varpi_4\left(a_D\sum_\alpha V^\alpha\rho^a\partial_\alpha - ia_I\sum_r\left[\mu_R-\mu_L\right]_a^r\partial_r\right)$

$= 2a_F\left(d\left(*\overline{\mathcal{F}}_A^a\right) - \varpi_4\left(\sum_\beta\left[\grave{A}_\beta,\mathcal{F}_A^{\alpha\beta}\right]^a\partial_\alpha\right)\right)$

and :

$\varpi_4 \left(\sum_\beta \left[\grave{A}_\beta, \mathcal{F}_A^{\alpha\beta} \right]^a \partial_\alpha \right)$
$= \sum_\alpha (-1)^{\alpha+1} \sum_\beta \left[\grave{A}_\beta, \mathcal{F}_A^{\alpha\beta} \right]^a \det O' dx^0 \wedge ..\widehat{dx^\alpha}.. \wedge dx^3$
$= \{- \sum_\beta \left[\grave{A}_\beta, \mathcal{F}_A^{0\beta} \right]^a dx^1 \wedge dx^2 \wedge dx^3 + \sum_\beta \left[\grave{A}_\beta, \mathcal{F}_A^{1\beta} \right]^a dx^0 \wedge dx^2 \wedge dx^3$
$- \sum_\beta \left[\grave{A}_\beta, \mathcal{F}_A^{2\beta} \right]^a dx^0 \wedge dx^1 \wedge dx^3 + \sum_\beta \left[\grave{A}_\beta, \mathcal{F}_A^{3\beta} \right]^a dx^0 \wedge dx^1 \wedge dx^2\} (\det O')$
On the other hand :
$\left[\grave{A}, *\overline{\mathcal{F}}_A \right]^a = \sum_{\lambda<\mu} \sum_\beta \left[\grave{A}_\beta, \left(*\overline{\mathcal{F}}_A \right)_{\lambda\mu} \right]^a dx^\beta \wedge dx^\lambda \wedge dx^\mu$
with :
$*\overline{\mathcal{F}}_A^a = -\{\mathcal{F}^{a,01} dx^3 \wedge dx^2 + \mathcal{F}^{a,02} dx^1 \wedge dx^3 + \mathcal{F}^{a,03} dx^2 \wedge dx^1$
$+ \mathcal{F}^{a,32} dx^0 \wedge dx^1 + \mathcal{F}^{a,13} dx^0 \wedge dx^2 + \mathcal{F}^{a,21} dx^0 \wedge dx^3\}$
$\left[\grave{A}, *\overline{\mathcal{F}}_A \right]^a =$
$- \sum_\beta \left[\grave{A}_\beta, \mathcal{F}^{01} \right]^a dx^\beta \wedge dx^3 \wedge dx^2 + \left[\grave{A}_\beta, \mathcal{F}^{02} \right]^a dx^\beta \wedge dx^1 \wedge dx^3 + \left[\grave{A}_\beta, \mathcal{F}^{03} \right]^a dx^\beta \wedge dx^2 \wedge dx^1$
$+ \left[\grave{A}_\beta, \mathcal{F}^{32} \right]^a dx^\beta \wedge dx^0 \wedge dx^1 + \left[\grave{A}_\beta, \mathcal{F}^{13} \right]^a dx^\beta \wedge dx^0 \wedge dx^2 + \left[\grave{A}_\beta, \mathcal{F}^{21} \right]^a dx^\beta \wedge dx^0 \wedge dx^3$
$= -\{- \left(\left[\grave{A}_1, \mathcal{F}^{01} \right]^a + \left[\grave{A}_2, \mathcal{F}^{02} \right]^a + \left[\grave{A}_3, \mathcal{F}^{03} \right]^a \right) dx^1 \wedge dx^2 \wedge dx^3$
$+ \left(\left[\grave{A}_0, \mathcal{F}^{10} \right]^a + \left[\grave{A}_3, \mathcal{F}^{13} \right]^a + \left[\grave{A}_2, \mathcal{F}^{12} \right]^a \right) dx^0 \wedge dx^2 \wedge dx^3\}$
$- \left(\left[\grave{A}_0, \mathcal{F}^{20} \right]^a + \left[\grave{A}_3, \mathcal{F}^{23} \right]^a + \left[\grave{A}_1, \mathcal{F}^{21} \right]^a \right) dx^0 \wedge dx^1 \wedge dx^3$
$+ \left(\left[\grave{A}_0, \mathcal{F}^{30} \right]^a + \left[\grave{A}_2, \mathcal{F}^{32} \right]^a + \left[\grave{A}_1, \mathcal{F}^{31} \right]^a \right) dx^0 \wedge dx^1 \wedge dx^2\}$
$= - \sum_\alpha (-1)^{\alpha+1} \left(\sum_\beta \left[\grave{A}_\beta, \mathcal{F}_A^{\alpha\beta} \right]^a \right) dx^0 \wedge ... \wedge \widehat{dx^\alpha} \wedge ... \wedge dx^3$
So : $\left[\grave{A}, *\overline{\mathcal{F}}_A \right]^a = -\varpi_4 \left(\sum_\beta \left[\grave{A}_\beta, \mathcal{F}_A^{\alpha\beta} \right]^a \partial_\alpha \right)$
Thus : $N\varpi_4 \left(a_D \sum_\alpha V^\alpha \rho^a \partial_\alpha - i a_I \sum_r [\mu_A]_a^r \partial_r \right) = d\left(*\overline{\mathcal{F}}_A^a \right) + \left[\grave{A}, *\overline{\mathcal{F}}_A \right]^a$
where we recognize the exterior covariant derivative.

And we have the geometric form of the equation :

$$\mathbf{N}\varpi_4 \left(a_D \rho_a \overrightarrow{V} - i a_I \overrightarrow{\mu}_a \right) = \nabla_e * \overline{\mathcal{F}}_A^a \qquad (95)$$

On the right hand side we have the moments, evaluated at a point through f, and at the right hand side the curvature form evaluated without f. Notice that the tensor ψ does not appear per se.

We see that the real part of the field acts through the "charge current" and the imaginary part through the "magnetic moment". So one can guess that the first impacts the velocity and the second the "rotation" of the particle.

15.3 Energy-momentum tensor

According to equations established in the previous part, the energy-momentum tensor $\delta^\alpha_\beta L$ can be expressed in two different equivalent ways from the moments and the force fields. We have a general, simple, equation which links all the force fields connections. It is then possible to get a simple equation for the scalar curvature, which does not require the explicit computation of the gravitational 2-form $\mathcal{F}_G$.

15.3.1 Moments

Now we have defined all the moments that we needed. As they are crucial in all the calculations it is convenient to sum up here their definitions and properties. They are all real scalar functions, invariant by gauge or chart changes.

1) The kinematic moments :

a) the "linear momentum" P : $P_a = \operatorname{Im} Tr\left([\psi^*][\gamma_0][\kappa_a][\psi]\right) = \operatorname{Im}\langle\psi, [\kappa_a][\psi]\rangle$

$\langle\psi, [\kappa_a][\psi]\rangle = Tr\left([\psi^*][\gamma_0][\kappa_a][\psi]\right) = iP_a$

$a < 4 : P_a = -i\frac{1}{2}Tr\left([\psi^*_R][\sigma_a][\psi_L] - [\psi^*_L][\sigma_a][\psi_R]\right);$

$a > 3 : P_a = -\frac{1}{2}Tr\left([\psi^*_R][\sigma_{a-3}][\psi_L] + [\psi^*_L][\sigma_{a-3}][\psi_R]\right)$

It depends only on 3 complex scalars :

a=1,2,3 : $p_a = Tr\left([\psi^*_R][\sigma_a][\psi_L]\right); \overline{p}_a = Tr\left([\psi^*_L][\sigma_a][\psi_R]\right)$

a<4 :$P_a = -i\frac{1}{2}\left(p^a - \overline{p}^a\right)$

a>3 :$P_a = -\frac{1}{2}\left(p^{a-3} + \overline{p}^{a-3}\right)$

It is never null :

$\sum_{a=1}^6 (P_a)^2 = \sum_{a=1}^3 \left|Tr\left([\psi^*_R][\sigma_a][\psi_L]\right)\right|^2 > 0$

The physical quantity is the tensor :$\sum_{a\alpha} V^\alpha P_a \partial_\alpha \otimes \overrightarrow{\kappa}_a$

b) the angular momentum J : $J_r = -\frac{1}{2}Tr\left(\eta^{rr}\psi^*_R\sigma_r\psi_R - \psi^*_L\sigma_r\psi_L\right)$

It can be null :

$\sum_{k=0}^3 J_k^2$

$= \frac{1}{4}\left(\sum_{k=0}^{3}\left(Tr\left(\psi_R^*\sigma_k\psi_R\right)\right)^2 + \left(Tr\left(\psi_L^*\sigma_k\psi_L\right)\right)^2 - 2\eta^{kk}Tr\left(\psi_R^*\sigma_k\psi_R\right)Tr\left(\psi_L^*\sigma_k\psi_L\right)\right)$

$\langle\psi,\gamma^r\psi\rangle = Tr\left([\psi]^*[\gamma_0\gamma^r][\psi]\right) = iTr\left(\eta^{rr}\psi_R^*\sigma_r\psi_R - \psi_L^*\sigma_r\psi_L\right) = -2iJ_r$

$\operatorname{Im}\langle\psi,[\gamma^r][\kappa_a]\psi\rangle = \operatorname{Im}Tr\left([\psi]^*[\gamma_0\gamma^r][\kappa_a][\psi]\right) = \left([J][\widetilde{\kappa}_a]\right)_r$

The physical quantity is the tensor :

$\sum_{a,r}\operatorname{Im}Tr\left([\psi]^*[\gamma_0\gamma^r][\kappa_a][\psi]\right)\partial_r\otimes\overrightarrow{\kappa}_a = \sum_{a,r}\left([J][\widetilde{\kappa}_a]\right)_r\partial_r\otimes\overrightarrow{\kappa}_a$ where $[J]$ is a 1x4 row matrix

c) one can add the function : $\langle\psi,\psi\rangle = -2\operatorname{Im}Tr\left([\psi_R]^*[\psi_L]\right) = iTr\left([\psi_R]^*[\psi_L] - [\psi_L]^*[\psi_R]\right)$

d) There is an important property of the partial derivative.

As it is easy to check the derivation commutes with the trace operator

$Tr\left(\partial_\beta\left([\psi_R]^*\sigma_r[\psi_R]\right)\right) = \sum_{j=1}^{m}\partial_\beta\left([\psi_R]^*\sigma_r[\psi_R]\right)_j^j$

$= \partial_\beta\sum_{j=1}^{m}\left([\psi_R]^*\sigma_r[\psi_R]\right)_j^j = \partial_\beta Tr\left([\psi_R]^*\sigma_r[\psi_R]\right)$

b) so :

$\partial_\beta Tr\left([\psi_R]^*\sigma_r[\psi_R]\right) = Tr\left(\partial_\beta\left([\psi_R]^*\sigma_r[\psi_R]\right)\right)$

$= Tr\left([\partial_\beta\psi_R]^*\sigma_r[\psi_R]\right) + Tr\left([\psi_R]^*\sigma_r[\partial_\beta\psi_R]\right)$

but :

$\overline{Tr\left([\partial_\beta\psi_R]^*\sigma_r[\psi_R]\right)} = Tr\left(\left([\partial_\beta\psi_R]^*\sigma_r[\psi_R]\right)^*\right) = Tr\left([\psi_R]^*\sigma_r[\partial_\beta\psi_R]\right)$

so :

$\partial_\beta Tr\left([\psi_R]^*\sigma_r[\psi_R]\right)$

$= Tr\left([\psi_R]^*\sigma_r[\partial_\beta\psi_R]\right) + \overline{Tr\left([\psi_R]^*\sigma_r[\partial_\beta\psi_R]\right)}$

$= 2\operatorname{Re}Tr\left([\psi_R]^*\sigma_r[\partial_\beta\psi_R]\right)$

$\Rightarrow \operatorname{Re}Tr\left([\psi_R]^*\sigma_r[\partial_\beta\psi_R]\right) = \frac{1}{2}Tr\left(\partial_\beta\left([\psi_R]^*\sigma_r[\psi_R]\right)\right)$

and we have the identity :

$\operatorname{Re}Tr\left(\eta^{rr}[\psi_R]^*\sigma_r[\partial_\beta\psi_R] - [\psi_L]^*\sigma_r[\partial_\beta\psi_L]\right)$

$= \frac{1}{2}\partial_\beta\operatorname{Re}Tr\left(\left(\eta^{rr}[\psi_R]^*\sigma_r[\psi_R] - [\psi_L]^*\sigma_r[\psi_L]\right)\right)$

$= \frac{1}{2}\partial_\beta Tr\left(\left(\eta^{rr}[\psi_R]^*\sigma_r[\psi_R] - [\psi_L]^*\sigma_r[\psi_L]\right)\right) = -\partial_\beta J_r$

And :

$\langle\psi,\gamma^r\partial_\beta\psi\rangle = Tr\left([\psi]^*[\gamma_0\gamma^r][\partial_\beta\psi]\right) = iTr\left(\eta^{rr}[\psi_R]^*\sigma_r[\partial_\beta\psi_R] - [\psi_L]^*\sigma_r[\partial_\beta\psi_L]\right)$

So

$\operatorname{Im}\langle\psi,\gamma^r\partial_\beta\psi\rangle = \operatorname{Im}Tr\left([\psi]^*[\gamma_0\gamma^r][\partial_\beta\psi]\right)$

$= \operatorname{Re}Tr\left(\eta^{rr}[\psi_R]^*\sigma_r[\partial_\beta\psi_R] - [\psi_L]^*\sigma_r[\partial_\beta\psi_L]\right) = -\partial_\beta J_r$

2) The force fields moments :

a) the "charges" $\rho_a = Tr\left([\psi_R]^*[\psi_L][\theta_a]^t - [\psi_L]^*[\psi_R][\theta_a]^t\right)$

$\left\langle \psi, [\psi][\theta_a]^t \right\rangle = Tr\left([\psi^*][\gamma_0][\psi][\theta_a]^t\right) = iTr\left([\psi_R]^*[\psi_L][\theta_a]^t - [\psi_L]^*[\psi_R][\theta_a]^t\right) = i\rho_a$

and the "charge current" : linked with the velocity : $\rho_a \overrightarrow{V} = \sum_\alpha V^\alpha \rho_a \partial_\alpha$

b) the magnetic moment : $[\mu_A]^r_a = -\left\langle \psi, \gamma^r [\psi][\theta_a]^t \right\rangle = -Tr\left([\psi]^*[\gamma_0\gamma^r][\psi][\theta_a]^t\right)$
$= -iTr\left(\eta^{rr}[\psi_R]^*\sigma_r[\psi_R][\theta_a]^t - [\psi_L]^*\sigma_r[\psi_L][\theta_a]^t\right)$

and the "magnetic moment" : linked with the tetrad : $\overrightarrow{\mu}_a = \sum_r [\mu_A]^r_a \partial_r$

3) The state tensor is the sum of 2 right and left components : $\psi = \psi_R + \psi_L$. If each of these components is decomposable : $\psi = \psi_R \otimes \sigma_R + \psi_L \otimes \sigma_L$ one can write the matrix

$[\psi] = \begin{bmatrix} \psi_R \\ \psi_L \end{bmatrix} = \begin{bmatrix} [\phi_R][\sigma_R] \\ [\phi_L][\sigma_L] \end{bmatrix}$

where $[\phi_R], [\phi_L]$ are 2x1 column matrices and $[\sigma_R], [\sigma_L]$ are 1xm row matrices. The previous formulas are simpler.

with any $[\mu]$ matrix : $Tr([\psi_1^*][\mu][\psi_2]) = Tr([\sigma_1]^*[\phi_1^*][\mu][\phi_2][\sigma_2]) = ([\phi_1^*][\mu][\phi_2])\, Tr([\sigma_1]^*[\sigma_2]) = ([\phi_1^*][\mu][\phi_2])\left([\sigma_2]^t\overline{[\sigma_1]}\right)$

$Tr\left([\psi_1^*][\mu][\psi_2][\theta_a]^t\right) = Tr\left([\sigma_1]^*[\phi_1^*][\mu][\phi_2][\sigma_2][\theta_a]^t\right) = ([\phi_1^*][\mu][\phi_2])\left([\sigma_2][\theta_a]^t[\sigma_1]^*\right)$

a) $P_a = \operatorname{Im} Tr([\psi^*][\gamma_0][\kappa_a][\psi])$:

$a < 4 : P_a = \operatorname{Im}\left(([\phi_R^*][\sigma_a][\phi_L])\left([\sigma_L]^t\overline{[\sigma_R]}\right)\right)$

$a > 3 : P_a = -\operatorname{Re}\left(([\phi_R^*][\sigma_{a-3}][\phi_L])\left([\sigma_L]^t\overline{[\sigma_R]}\right)\right)$

b) $J_r = -\frac{1}{2}Tr\left(\eta^{rr}\psi_R^*\sigma_r\psi_R - \psi_L^*\sigma_r\psi_L\right)$
$= -\frac{1}{2}\left(\eta^{rr}([\phi_R^*][\sigma_a][\phi_R])\left([\sigma_R]^t\overline{[\sigma_R]}\right) - ([\phi_L^*][\sigma_a][\phi_L])\left([\sigma_L]^t\overline{[\sigma_L]}\right)\right)$

c) $\langle \psi, \psi \rangle = -2\operatorname{Im} Tr([\psi_R]^*[\psi_L]) = -2\operatorname{Im}\left(([\phi_R]^*[\phi_L])\left([\sigma_L]^t\overline{[\sigma_R]}\right)\right)$

d) $\rho^a = Tr\left([\psi_R]^*[\psi_L][\theta_a]^t - [\psi_L]^*[\psi_R][\theta_a]^t\right)$
$= ([\phi_R^*][\phi_L])\left([\sigma_L][\theta_a]^t[\sigma_R]^*\right) - ([\phi_L^*][\phi_R])\left([\sigma_R][\theta_a]^t[\sigma_L]^*\right)$

$\overline{\left([\sigma_L][\theta_a]^t[\sigma_R]^*\right)} = \left([\sigma_L][\theta_a]^t[\sigma_R]^*\right)^* = \left([\sigma_R]^*\overline{[\theta_a]}[\sigma_L]\right) = -\left([\sigma_R]^*[\theta_a]^t[\sigma_L]\right)$

$\overline{([\phi_R^*][\phi_L])} = ([\phi_R^*][\phi_L])^* = ([\phi_L^*][\phi_R])$

$\rho^a = 2\operatorname{Re}\left(([\phi_R^*][\phi_L])\left([\sigma_L][\theta_a]^t[\sigma_R]^*\right)\right)$

e) $[\mu_A]^r_a = -i\left(\eta^{rr}[\phi_R^*]\sigma_r[\phi_R]\left([\sigma_R][\theta_a]^t[\sigma_R]^*\right) - [\psi_L]^*\sigma_r[\psi_L][\theta_a]^t\left([\sigma_L][\theta_a]^t[\sigma_L]^*\right)\right)$

4) The matter lagrangian can be expressed with respect to the moments :

a) $\operatorname{Im}\langle\psi,\nabla_\alpha\psi\rangle=\operatorname{Im}\langle\psi,\partial_\alpha\psi\rangle+\sum_a G_\alpha^a\operatorname{Im}\langle\psi,[\kappa_a][\psi]\rangle+\operatorname{Im}\grave{A}_\alpha^a\left\langle\psi,[\psi][\theta_a]^t\right\rangle$

$\operatorname{Im}\langle\psi,\nabla_\alpha\psi\rangle=\operatorname{Im}\langle\psi,\partial_\alpha\psi\rangle+\sum_a G_\alpha^a P_a+\rho_a\operatorname{Re}\grave{A}_\alpha^a$

b) $\operatorname{Im}\langle\psi,\gamma^r\nabla_\alpha\psi\rangle=\operatorname{Im}\langle\psi,\gamma^r\partial_\alpha\psi\rangle+\sum_a G_\alpha^a\operatorname{Im}\langle\psi,\gamma^r[\kappa_a][\psi]\rangle+\operatorname{Im}\grave{A}_\alpha^a\left\langle\psi,\gamma^r[\psi][\theta_a]^t\right\rangle$

$\operatorname{Im}\langle\psi,\gamma^r\nabla_\alpha\psi\rangle=-\partial_\alpha J_r+\sum_a G_\alpha^a([J][\widetilde{\kappa}_a])_r-[\mu_R-\mu_L]_a^r\operatorname{Im}\grave{A}_\alpha^a$

c) $\operatorname{Im}\langle\psi,D_M^\alpha\nabla_\alpha\psi\rangle=\operatorname{Im}\langle\psi,\left(\sum_r a_I O_r^\alpha[\gamma^r]+\sum_\alpha V^\alpha a_D I\right)\nabla_\alpha\psi\rangle$

$\operatorname{Im}\langle\psi,D_M^\alpha\nabla_\alpha\psi\rangle=a_I\left(-\sum_{\alpha,r}O_r^\alpha\partial_\alpha J_r+\sum_{ar}G_r^a([J][\widetilde{\kappa}_a])_r-[\mu_R-\mu_L]_a^r\operatorname{Im}\grave{A}_r^a\right)+$
$a_D\sum_\alpha V^\alpha\left(\operatorname{Im}\langle\psi,\partial_\alpha\psi\rangle+\sum_a\left(G_\alpha^a P_a+\rho_a\operatorname{Re}\grave{A}_\alpha^a\right)\right)$

d) $L_M=a_M\langle\psi,\psi\rangle+a_I\left(-\sum_{\alpha,r}O_r^\alpha\partial_\alpha J_r+\sum_{ar}G_r^a([J][\widetilde{\kappa}_a])_r-[\mu_R-\mu_L]_a^r\operatorname{Im}\grave{A}_r^a\right)+$
$a_D\sum_\alpha V^\alpha\left(\operatorname{Im}\langle\psi,\partial_\alpha\psi\rangle+\sum_a\left(G_\alpha^a P_a+\rho_a\operatorname{Re}\grave{A}_\alpha^a\right)\right)$

Notice that when the sections are composed with f then the full derivatives must be used :

$\operatorname{Im}\langle\psi,\nabla_\alpha\psi\rangle=\operatorname{Im}\left\langle\psi,\frac{d\psi}{d\xi^\alpha}\right\rangle+\sum_a G_\alpha^a P_a+\rho_a\operatorname{Re}\grave{A}_\alpha^a$

$\operatorname{Im}\langle\psi,\gamma^r\nabla_\alpha\psi\rangle=-\frac{dJ_r}{d\xi^\alpha}+\sum_a G_\alpha^a([J][\widetilde{\kappa}_a])_r-[\mu_R-\mu_L]_a^r\operatorname{Im}\grave{A}_\alpha^a$

15.3.2 The energy-momentum tensor

Back to the energy momentum tensor. We have two ways to compute $\delta_\beta^\alpha L$ which will be both useful.

1) Equation 68 reads :

$\delta_\beta^\alpha L=-\frac{dNL_M}{dV^\beta}V^\alpha+\sum_{i,j}\left(\frac{dNL_M}{d\operatorname{Re}\nabla_\alpha\psi^{ij}}\operatorname{Re}\partial_\beta\psi^{ij}+\frac{dNL_M}{d\operatorname{Im}\nabla_\alpha\psi^{ij}}\operatorname{Im}\partial_\beta\psi^{ij}\right)$
$+2\sum_{a\gamma}\left(\frac{dL_F}{d\mathcal{F}_{G\alpha\gamma}^a}\partial_\beta G_\gamma^a+\frac{dL_F}{d\operatorname{Re}\mathcal{F}_{A,\alpha\gamma}^a}\operatorname{Re}\partial_\beta\grave{A}_\gamma^a+\frac{dL_F}{d\operatorname{Im}\mathcal{F}_{A,\alpha\gamma}^a}\operatorname{Im}\partial_\beta\grave{A}_\gamma^a\right)+\sum_{i\gamma}\frac{dL}{d\partial_\alpha O'^i_\gamma}\partial_\beta O'^i_\gamma$

$\frac{dL_M}{d\operatorname{Re}\nabla_\alpha\psi^{ij}}=N\operatorname{Im}\left([\psi]^*[\gamma_0][D_M^\alpha]\right)_i^j;\ \frac{dL_M}{d\operatorname{Im}\nabla_\alpha\psi^{ij}}=N\operatorname{Re}\left([\psi]^*[\gamma_0][D_M^\alpha]\right)_i^j$

$\frac{dL_F}{d\mathcal{F}_{G\alpha\gamma}^a}=a_G\left(O_{p_a}^\gamma O_{q_a}^\alpha-O_{q_a}^\gamma O_{p_a}^\alpha\right);\ \frac{dL_F}{d\operatorname{Re}\mathcal{F}_{A,\alpha\gamma}^a}=a_F\operatorname{Re}\mathcal{F}_A^{a,\alpha\gamma};\ \frac{dL_F}{d\operatorname{Im}\mathcal{F}_{A,\alpha\gamma}^a}=a_F\operatorname{Im}\mathcal{F}_A^{a,\alpha\gamma}$

$\frac{dL}{d\partial_\alpha O'^i_\gamma}=0$

$\frac{dNL_M}{dV^\beta}=a_D N\operatorname{Im}\langle\psi,\nabla_\beta\psi\rangle$

$\delta_\beta^\alpha L=-a_D NV^\alpha\operatorname{Im}\langle\psi,\nabla_\beta\psi\rangle+\sum_{i,j}\operatorname{Im}\left([\psi]^*[\gamma_0][D_M^\alpha]\right)_i^j\operatorname{Re}\partial_\beta\psi^{ij}$
$+\operatorname{Re}\left([\psi]^*[\gamma_0][D_M^\alpha]\right)_i^j\operatorname{Im}\partial_\beta\psi^{ij}+2\sum_{a,\gamma}a_G\left(O_{p_b}^\gamma O_{q_b}^\alpha-O_{q_b}^\gamma O_{p_b}^\alpha\right)\partial_\beta G_\gamma^a$
$+a_F\operatorname{Re}\mathcal{F}_A^{a,\alpha\gamma}\operatorname{Re}\partial_\beta\grave{A}_\gamma^a+a_F\operatorname{Im}\mathcal{F}_A^{a,\alpha\gamma}\operatorname{Im}\partial_\beta\grave{A}_\gamma^a$

$\delta_\beta^\alpha L=-a_D NV^\alpha\operatorname{Im}\langle\psi,\nabla_\beta\psi\rangle+N\operatorname{Im}\langle\psi,[\gamma_0][D_M^\alpha][\partial_\beta\psi]\rangle$
$+2\sum_{a,\gamma}a_G\left(O_{p_b}^\gamma O_{q_b}^\alpha-O_{q_b}^\gamma O_{p_b}^\alpha\right)\partial_\beta G_\gamma^a+a_F\left(\operatorname{Re}\mathcal{F}_A^{a,\alpha\gamma}\overline{\partial_\beta\grave{A}_\gamma^a}\right)$

So the equation becomes :

$$\delta_\beta^\alpha L = -N\left(a_I \sum_r O_r^\alpha \frac{dJ_r}{d\xi^\alpha} + a_D V^\alpha \sum_a \left(G_\beta^a P_a + \rho^a \operatorname{Re} \grave{A}_\beta^a\right)\right)$$
$$+2\sum_{a,\gamma}\left(a_G\left(O_{p_a}^\gamma O_{q_a}^\alpha - O_{q_a}^\gamma O_{p_a}^\alpha\right)\partial_\beta G_\gamma^a + a_F \operatorname{Re}\left(\partial_\beta \grave{A}_\gamma, \mathcal{F}_A^{\alpha\gamma}\right)\right) \tag{96}$$

2) We can get a more convenient equation. We go back to the equation 55 which reads here :

$\forall\alpha,\beta : 0 = \sum_{i,} \frac{dL}{dO_\alpha'^i} O_\beta'^i + \delta_\beta^\alpha L$

expressed with respect to O :

$\frac{dL}{dO_\alpha'^i} = \sum_{j\beta} \frac{dL}{dO_j^\beta}\frac{dO_j^\beta}{dO_\alpha'^i} = -\sum_{j\beta} O_i^\beta O_j^\alpha \frac{dL}{dO_j^\beta}$

a) Let us compute the derivatives :

$L_M = a_M \langle\psi,\psi\rangle + a_I \operatorname{Im}\sum_{\lambda r} O_r^\lambda \langle\psi,\gamma^r\nabla_\lambda\psi\rangle + a_D\sum_\alpha V^\alpha\langle\psi,\nabla_\alpha\psi\rangle$

$\frac{dL_M}{dO_j^\beta} = a_I \operatorname{Im}\langle\psi,\gamma^j\nabla_\beta\psi\rangle$

$\frac{dL_M}{dO_\alpha'^i} = -\sum_{j\beta} O_i^\beta O_j^\alpha a_I \operatorname{Im}\langle\psi,\gamma^j\nabla_\beta\psi\rangle$

$L_F = \frac{1}{2}a_F\sum_{a\lambda\mu\gamma\xi pqkl}\eta^{qp}\eta^{lk}O_q^\lambda O_p^\gamma O_l^\mu O_k^\xi \mathcal{F}_{A\gamma\xi}^a \overline{\mathcal{F}}_{A\lambda\mu}^a$

$+a_G\sum_{a,\lambda\mu}\mathcal{F}_{G\lambda\mu}^a\left(O_{p_a}^\mu O_{q_a}^\lambda - O_{q_a}^\mu O_{p_a}^\lambda\right)$

$\frac{\partial L_F}{\partial O_j^\beta} =$

$\frac{1}{2}a_F\sum\eta^{pq}\eta^{lk}\mathcal{F}_{A\gamma\xi}^a\overline{\mathcal{F}}_{A\lambda\mu}^a\left(\delta_\lambda^\beta\delta_q^j O_p^\gamma O_l^\mu O_k^\xi + O_q^\lambda\delta_\gamma^\beta\delta_p^j O_l^\mu O_k^\xi + O_q^\lambda O_p^\gamma\delta_\mu^\beta\delta_l^j O_k^\xi + O_q^\lambda O_p^\gamma O_l^\mu\delta_\xi^\beta\delta_k^j\right)$

$+a_G\sum_{a,\lambda\mu}\mathcal{F}_{G\lambda\mu}^a\left(\delta_\mu^\beta\delta_{p_a}^j O_{q_a}^\lambda + O_{p_a}^\mu\delta_\lambda^\beta\delta_{q_a}^j - \delta_\mu^\beta\delta_{q_a}^j O_{p_a}^\lambda - O_{q_a}^\mu\delta_\lambda^\beta\delta_{p_a}^j\right)$

$= \frac{1}{2}a_F\sum(\eta^{pj}g^{\mu\lambda}\mathcal{F}_{A\gamma\lambda}^a\overline{\mathcal{F}}_{A\beta\mu}^a O_p^\gamma + \eta^{jq}O_q^\lambda g^{\gamma\mu}\mathcal{F}_{A\beta\mu}^a\overline{\mathcal{F}}_{A\lambda\gamma}^a - g^{\mu\gamma}\eta^{jk}O_k^\lambda\mathcal{F}_{A\gamma\lambda}^a\overline{\mathcal{F}}_{A\beta\mu}^a$

$-g^{\lambda\mu}\eta^{lj}O_l^\gamma\mathcal{F}_{A\beta\mu}^a\overline{\mathcal{F}}_{A\lambda\gamma}^a) + 2a_G\sum_{a,\lambda\mu}\mathcal{F}_{G\beta\lambda}^a\left(O_{p_a}^\lambda\delta_{q_a}^j - O_{q_a}^\lambda\delta_{p_a}^j\right)$

$\frac{dL_F}{dO_\alpha'^i} = -\sum_{j\beta}O_i^\beta O_j^\alpha\frac{dL}{dO_j^\beta}$

$= -\frac{1}{2}a_F O_i^\beta\sum(\eta^{pj}O_p^\gamma O_j^\alpha g^{\mu\lambda}\mathcal{F}_{A\gamma\lambda}^a\overline{\mathcal{F}}_{A\beta\mu}^a + \eta^{jq}O_q^\lambda O_j^\alpha g^{\gamma\mu}\mathcal{F}_{A\beta\mu}^a\overline{\mathcal{F}}_{A\lambda\gamma}^a$

$-g^{\mu\gamma}\eta^{jk}O_k^\lambda O_j^\alpha\mathcal{F}_{A\gamma\lambda}^a\overline{\mathcal{F}}_{A\beta\mu}^a - g^{\lambda\mu}\eta^{lj}O_l^\gamma O_j^\alpha\mathcal{F}_{A\beta\mu}^a\overline{\mathcal{F}}_{A\lambda\gamma}^a) - 2a_G O_i^\beta\sum\mathcal{F}_{G\beta\lambda}^a\left(O_{p_a}^\lambda\delta_{q_a}^j O_j^\alpha - O_{q_a}^\lambda\delta_{p_a}^j O_j^\alpha\right)$

$\frac{dL_F}{dO_\alpha'^i} = -\frac{1}{2}a_F O_i^\beta\sum\left(\mathcal{F}_A^{a\alpha\gamma}\overline{\mathcal{F}}_{A\beta\gamma}^a + \mathcal{F}_{A\beta\gamma}^a\overline{\mathcal{F}}_A^{a\alpha\gamma} + \mathcal{F}_A^{a\alpha\gamma}\overline{\mathcal{F}}_{A\beta\gamma}^a + \mathcal{F}_{A\beta\gamma}^a\overline{\mathcal{F}}_A^{a\alpha\gamma}\right)$

$-2a_G O_i^\beta\sum\mathcal{F}_{G\beta\gamma}^a\left(O_{p_a}^\gamma O_{q_a}^\alpha - O_{q_a}^\gamma O_{p_a}^\alpha\right)$

$\frac{dL_F}{dO_\alpha'^i} = -2a_F\sum_\beta O_i^\beta\sum_\gamma\operatorname{Re}(\mathcal{F}_A^{\alpha\gamma},\mathcal{F}_{A\beta\gamma}) - 2a_G O_i^\beta\sum_{a\gamma}\mathcal{F}_{G\beta\gamma}^a\left(O_{p_a}^\gamma O_{q_a}^\alpha - O_{q_a}^\gamma O_{p_a}^\alpha\right)$

$\frac{dL}{dO_\alpha'^i}$

$= -\sum_\beta O_i^\beta a_I \sum_r O_r^\alpha \operatorname{Im}\langle\psi,\gamma^r\nabla_\beta\psi\rangle$
$+2a_F\sum_\gamma \operatorname{Re}\left(\mathcal{F}_A^{\alpha\gamma},\mathcal{F}_{A\beta\gamma}\right)+2a_G\sum_{a\gamma}\mathcal{F}_{G\beta\gamma}^a\left(O_{p_a}^\gamma O_{q_a}^\alpha - O_{q_a}^\gamma O_{p_a}^\alpha\right)$
b) Thus :

$$\forall\alpha,\beta:\delta_\beta^\alpha L = a_I N \operatorname{Im}\langle\psi,\gamma^\alpha\nabla_\beta\psi\rangle + 2a_F\sum_\gamma \operatorname{Re}\left(\mathcal{F}_A^{\alpha\gamma},\mathcal{F}_{A\beta\gamma}\right) +2a_G\sum_{a\lambda}\mathcal{F}_{G\beta\lambda}^a\left(O_{p_a}^\gamma O_{q_a}^\alpha - O_{q_a}^\gamma O_{p_a}^\alpha\right) \tag{97}$$

3) Using the moments we get for equation 97 :
$\forall\alpha,\beta : \delta_\beta^\alpha L = a_I N\sum_{\alpha r} O_r^\alpha \operatorname{Im}\langle\psi,\gamma^r\nabla_\beta\psi\rangle + 2a_F\sum_\gamma \operatorname{Re}\left(\mathcal{F}_A^{\alpha\gamma},\mathcal{F}_{A\beta\gamma}\right) + 2a_G\sum_{a\lambda}\mathcal{F}_{G\beta\lambda}^a\left(O_{p_a}^\gamma O_{q_a}^\alpha - O_{q_a}^\gamma O_{p_a}^\alpha\right)$

$$\forall\alpha,\beta:\delta_\beta^\alpha L = N a_I\sum_r O_r^\alpha\left(-\frac{dJ_r}{d\xi^\beta}+\sum_a\left(G_\beta^a\left([J]\left[\widetilde{\kappa}_a\right]\right)_r - \left[\mu_A\right]_a^r \operatorname{Im}\grave{A}_\beta^a\right)\right) +2a_F\sum_\gamma \operatorname{Re}\left(\mathcal{F}_A^{\alpha\gamma},\mathcal{F}_{A\beta\gamma}\right)+2a_G\sum_{a\lambda}\mathcal{F}_{G\beta\lambda}^a\left(O_{p_a}^\gamma O_{q_a}^\alpha - O_{q_a}^\gamma O_{p_a}^\alpha\right) \tag{98}$$

15.3.3 Superpotential

The superpotential is here :
$\Pi_{H\beta} = \sum_a\left(\operatorname{Re}\grave{A}_\beta^a\right)\Pi_{AR}^a + \left(\operatorname{Im}\grave{A}_\beta^a\right)\Pi_{AI}^a + a_G G_\beta^a \Pi_G^a$
$\Pi_G^a = 4\left(\det O'\right)\sum_{\lambda<\mu}\sum_{\alpha<\gamma}\epsilon(\lambda,\mu,\alpha,\gamma)\left(O_{p_a}^\gamma O_{q_a}^\alpha - O_{q_a}^\gamma O_{p_a}^\alpha\right)dx^\lambda\wedge dx^\mu$
$\Pi_{AR}^a = 4a_F\left(\det O'\right)\sum_{\lambda<\mu}\sum_{\alpha<\gamma}\epsilon(\lambda,\mu,\alpha,\gamma)\operatorname{Re}\mathcal{F}_A^{a,\alpha\beta}dx^\lambda\wedge dx^\mu$
$\Pi_{AI}^a = 4a_F\left(\det O'\right)\sum_{\lambda<\mu}\sum_{\alpha<\gamma}\epsilon(\lambda,\mu,\alpha,\gamma)\operatorname{Im}\mathcal{F}_A^{a,\alpha\gamma}dx^\lambda\wedge dx^\mu$
$\left(\operatorname{Re}\grave{A}_\beta^a\right)\Pi_{AR}^a + \left(\operatorname{Im}\grave{A}_\beta^a\right)\Pi_{AI}^a$
$= 4a_F\left(\det O'\right)\sum_{\lambda<\mu}\sum_{\alpha<\gamma}\epsilon(\lambda,\mu,\alpha,\gamma)\left(\operatorname{Re}\mathcal{F}_A^{a,\alpha\gamma}\left(\operatorname{Re}\grave{A}_\beta^a\right)+\left(\operatorname{Im}\grave{A}_\beta^a\right)\operatorname{Im}\mathcal{F}_A^{a,\alpha\gamma}\right)dx^\lambda\wedge dx^\mu$
$= 4a_F\left(\det O'\right)\sum_{\lambda<\mu}\sum_{\alpha<\gamma,a}\epsilon(\lambda,\mu,\alpha,\gamma)\operatorname{Re}\left(\overline{\grave{A}}_\beta^a\mathcal{F}_A^{a,\alpha\gamma}\right)dx^\lambda\wedge dx^\mu$
and :

(99)

$\Pi_{H\beta} = 4 \det O' \sum_a \sum_{\substack{\lambda<\mu \\ \alpha<\gamma}}$
$\epsilon(\lambda,\mu,\alpha,\gamma) \left(a_G G^a_\beta \left(O^\gamma_{p_a} O^\alpha_{q_a} - O^\gamma_{q_a} O^\alpha_{p_a} \right) + a_F \operatorname{Re}\left(\overline{\grave{A}}^a_\beta \mathcal{F}^{a,\alpha\gamma}_A \right) \right) dx^\lambda \wedge dx^\mu$

15.3.4 The equation

The equation 66 reads here : $d(\Pi_{H\beta}) = 0$

That is :

$d\Pi_{H\beta}$

$= -4 \sum_a \varpi_4 \left(\sum_{\alpha\gamma} \partial_\gamma \left(\sum_a \left(\begin{array}{c} a_G G^a_\beta \left(O^\gamma_{p_a} O^\alpha_{q_a} - O^\gamma_{q_a} O^\alpha_{p_a} \right) \\ + a_F \operatorname{Re}\left(\overline{\grave{A}}^a_\beta \mathcal{F}^{a,\alpha\gamma}_A \right) \end{array} \right) \det O' \right) \partial_\alpha \right)$

$= -4 \sum_{\alpha\gamma} (-1)^{\alpha+1} \partial_\gamma \left(\sum_a \left(\begin{array}{c} a_G G^a_\beta \left(O^\gamma_{p_a} O^\alpha_{q_a} - O^\gamma_{q_a} O^\alpha_{p_a} \right) \\ + a_F \operatorname{Re}\left(\overline{\grave{A}}^a_\beta \mathcal{F}^{a,\alpha\gamma}_A \right) \end{array} \right) \det O' \right)$

$\times dx^0 \wedge ..\widehat{dx^\alpha} \wedge ... \wedge dx^3$

So :

$$\forall \alpha, \beta : \mathbf{0} = \sum_\gamma \partial_\gamma \left(\left(a_F \operatorname{Re}\left(\grave{A}_\beta, \mathcal{F}^{\alpha\gamma}_A \right) + a_G \sum_a \left(O^\gamma_{p_a} O^\alpha_{q_a} - O^\gamma_{q_a} O^\alpha_{p_a} \right) G^a_\beta \right) \det O' \right) \tag{100}$$

One can check that this equation is equivalent to the equality of the two previous expressions for $\delta^\alpha_\beta L$.This equation involves neither f (all functions come from L_F) or the state tensor ψ.A change in the gravitational field should entail a change in the other fields, whatever the presence of particles, and conversely.

15.3.5 Scalar curvature

If we put $\alpha = \beta$ in equation 97 we get :

$\forall \alpha : L = N a_I \operatorname{Im} \langle \psi, \gamma^\alpha \nabla_\alpha \psi \rangle + 2 a_F \sum_\lambda \operatorname{Re}\left(\mathcal{F}_{A\alpha\lambda}, \mathcal{F}^{\alpha\lambda}_A \right)$
$+ 2 a_G \sum_{a\lambda} \mathcal{F}^a_{G\alpha\lambda} \left(O^\alpha_{q_a} O^\lambda_{p_a} - O^\alpha_{p_a} O^\lambda_{q_a} \right)$

and by adding over α :

$4L = N a_I \operatorname{Im} \langle \psi, D\psi \rangle + 2 a_F \sum_{\alpha\lambda} \left(\mathcal{F}^a_{A\alpha\lambda}, \mathcal{F}^{a\alpha\lambda}_A \right)$
$+ 2 a_G \sum_{\alpha\lambda a} \mathcal{F}^a_{G\alpha\lambda} \left(O^\alpha_{q_a} O^\lambda_{p_a} - O^\alpha_{p_a} O^\lambda_{q_a} \right)$

That is :

$4L = N a_I \operatorname{Im} \langle \psi, D\psi \rangle + 4 a_F \langle \mathcal{F}_A, \mathcal{F}_A \rangle + 2R$

with :
$a_F \langle \mathcal{F}_A, \mathcal{F}_A \rangle = a_F \sum_a \sum_{\{\alpha\beta\}} \overline{\mathcal{F}}^a_{A\alpha\beta} \mathcal{F}_A^{a,\alpha\beta}$
$= \frac{1}{2} a_F \sum_a \sum_{\alpha\beta} \overline{\mathcal{F}}^a_{A\alpha\beta} \mathcal{F}_A^{a,\alpha\beta} = \frac{1}{2} a_F \sum_{\alpha\beta} \left(\mathcal{F}_{A\alpha\beta}, \mathcal{F}_A^{\alpha\beta} \right)$
$a_G R = a_G \sum_{\alpha\lambda} \mathcal{F}^a_{G\alpha\lambda} \left(O^\alpha_{q_a} O^\lambda_{p_a} - O^\alpha_{p_a} O^\lambda_{q_a} \right)$
But :
$L = N a_M \langle \psi, \psi \rangle + N a_I \operatorname{Im} \langle \psi, D\psi \rangle$
$+ N a_D \sum_\alpha \operatorname{Im} \langle \psi, V^\alpha \nabla_\alpha \rangle + a_F \langle \mathcal{F}_A, \mathcal{F}_A \rangle + a_G R$
So equation 97 implies :
$N a_I \operatorname{Im} \langle \psi, D\psi \rangle + 4 a_F \langle \mathcal{F}_A, \mathcal{F}_A \rangle + 2R = 4 N a_M \langle \psi, \psi \rangle + 4 N a_I \operatorname{Im} \langle \psi, D\psi \rangle$
$+ 4 N a_D \sum_\alpha \operatorname{Im} \langle \psi, V^\alpha \nabla_\alpha \rangle + 4 a_F \langle \mathcal{F}_A, \mathcal{F}_A \rangle + 4 a_G R$
So we get the formula for the scalar curvature :

$$R = -\frac{N}{a_G} \left(2 a_M \langle \psi, \psi \rangle + \frac{3}{2} a_I \operatorname{Im} \langle \psi, D\psi \rangle + 2 a_D \sum_\alpha \operatorname{Im} \langle \psi, V^\alpha \nabla_\alpha \rangle \right) \quad (101)$$

It depends on the particles and the other fields, as expected, but it is null if there is no particle. The curvature being small, the constant a_G should be much larger than a_M, a_I, a_D. As the sign of N is difficult to predict one cannot guess anything about the signs of the terms. All the second order (the torsion and the scalar curvature) gravitational quantities are thus easily computed, without need of the curvature form $\mathcal{F}_G$ which does not appear in the equations.

15.4 Equation of state

The equations of state is written from the results of the previous part. It is not too complicated and involves the derivatives of the velocity. But from it one can prove two striking conservation laws for the moments: one related to the density of particles and the other to the "particles energy":
$\sum_\alpha \frac{d}{d\xi^\alpha} \left((N \det O') V^\alpha \langle \psi, \psi \rangle \right) = 0$
$N (\det O') L_M + a_I \sum_{\alpha r} \frac{d}{d\xi^\alpha} \left(N (\det O') O^\alpha_r J_r \right) = 0$

15.4.1 Equation

The equations 50,51 reads here :
$\forall i, j : 0 = N \{ \sum_{\alpha k} \frac{d L_M^\diamondsuit}{d \operatorname{Re} \nabla_\alpha \psi^{kj}} \operatorname{Re} [G_\alpha]^k_i + \frac{d L_M^\diamondsuit}{d \operatorname{Im} \nabla_\alpha \psi^{kj}} \operatorname{Im} [G_\alpha]^k_i$

$+\frac{dL_M^\diamondsuit}{d\,\mathrm{Re}\,\nabla_\alpha\psi^{ik}}\,\mathrm{Re}\left[\grave{A}_\alpha\right]_j^k+\frac{dL_M^\diamondsuit}{d\,\mathrm{Im}\,\nabla_\alpha\psi^{ik}}\,\mathrm{Im}\left[\grave{A}_\alpha\right]_j^k\}$

$-\sum_\beta\frac{1}{(\det O')}\frac{d}{d\xi^\beta}\left(\frac{d\mathcal{L}_M}{d\,\mathrm{Re}\,\nabla_\beta\psi^{ij}}\right)+N\frac{\partial L_M^\diamondsuit}{\partial\,\mathrm{Re}\,\psi^{ij}}$

$\forall i,j:0=N\{\sum_{\alpha k}-\frac{dL_M^\diamondsuit}{d\,\mathrm{Re}\,\nabla_\alpha\psi^{kj}}\,\mathrm{Im}\,[G_\alpha]_i^k+\frac{dL_M^\diamondsuit}{d\,\mathrm{Im}\,\nabla_\alpha\psi^{kj}}\,\mathrm{Re}\,[G_\alpha]_i^k$

$-\frac{dL_M^\diamondsuit}{d\,\mathrm{Re}\,\nabla_\alpha\psi^{ik}}\,\mathrm{Im}\left[\grave{A}_\alpha\right]_j^k+\frac{dL_M^\diamondsuit}{d\,\mathrm{Im}\,\nabla_\alpha\psi^{ik}}\,\mathrm{Re}\left[\grave{A}_\alpha\right]_j^k$

$-\sum_\beta\frac{1}{(\det O')}\frac{d}{d\xi^\beta}\left(\frac{d\mathcal{L}_M}{d\,\mathrm{Im}\,\nabla_\alpha\psi^{ij}}\right)+N\frac{\partial L_M^\diamondsuit}{\partial\,\mathrm{Im}\,\psi^{ij}}$

1) Computation of the derivatives $\frac{\partial L_M^\diamondsuit}{\partial\,\mathrm{Re}\,\psi^{ij}},\frac{\partial L_M^\diamondsuit}{\partial\,\mathrm{Im}\,\psi^{ij}}$. Notice that only the terms in ψ (and not $\nabla_\alpha\psi$) are involved here.

$L_M=N\left(a_M\langle\psi,\psi\rangle+\mathrm{Im}\,\langle\psi,\sum_\alpha D_M^\alpha\left[\nabla_\alpha\psi\right]\rangle\right)$

$\langle\psi,\psi\rangle=\sum_{pqk}\overline{[\psi]}^{kp}\left[\gamma_0\right]_q^k[\psi]^{qp}$

$\frac{\partial\langle\psi,\psi\rangle}{\partial\,\mathrm{Re}\,\psi^{ij}}=\sum\delta_k^i\delta_p^j\left[\gamma_0\right]_q^k[\psi]^{qp}+\overline{[\psi]}^{kp}\left[\gamma_0\right]_q^k\delta_q^i\delta_p^j$

$=\sum_{pqk}\left[\gamma_0\right]_q^i[\psi]^{qj}+\overline{[\psi]}^{kj}\left[\gamma_0\right]_i^k=\left(\left[\gamma_0\right]\left([\psi]-\overline{[\psi]}\right)\right)_j^i$

$\frac{\partial\langle\psi,\psi\rangle}{\partial\,\mathrm{Re}\,\psi^{ij}}=2\left(i\left[\gamma_0\right]\mathrm{Im}\,[\psi]\right)_j^i=2\,\mathrm{Im}\left(i\left[\gamma_0\right][\psi]\right)_j^i$

$\frac{\partial\langle\psi,\psi\rangle}{\partial\,\mathrm{Im}\,\psi^{ij}}=\sum-i\delta_k^i\delta_p^j\left[\gamma_0\right]_q^k[\psi]^{qp}+i\,\overline{[\psi]}^{kp}\left[\gamma_0\right]_q^k\delta_q^i\delta_p^j$

$=\sum_{pqk}-i\left[\gamma_0\right]_q^i[\psi]^{qj}+i\,\overline{[\psi]}^{kj}\left[\gamma_0\right]_i^k=-i\left(\left[\gamma_0\right]\left([\psi]+\overline{[\psi]}\right)\right)_j^i$

$\frac{\partial\langle\psi,\psi\rangle}{\partial\,\mathrm{Im}\,\psi^{ij}}=-2i\left(\left[\gamma_0\right]\mathrm{Re}\,[\psi]\right)_j^i=-2\,\mathrm{Re}\left(i\left[\gamma_0\right][\psi]\right)_j^i$

$\mathrm{Im}\,\langle\psi,\sum_\alpha D_M^\alpha\left[\nabla_\alpha\psi\right]\rangle=\sum_{\alpha pq}\mathrm{Im}\left(\overline{[\psi]}^{qp}\left(\left[\gamma_0\right]\left[D_M^\alpha\right]\left[\nabla_\alpha\psi\right]\right)^{qp}\right)$

$=\sum_{\alpha pq}\mathrm{Im}\left(\overline{[\psi]}^{qp}\right)\mathrm{Re}\left(\left[\gamma_0\right]\left[D_M^\alpha\right]\left[\nabla_\alpha\psi\right]\right)^{qp}+\mathrm{Re}\left(\overline{[\psi]}^{qp}\right)\mathrm{Im}\left(\left[\gamma_0\right]\left[D_M^\alpha\right]\left[\nabla_\alpha\psi\right]\right)^{qp}$

$=\sum_{\alpha pq}-\mathrm{Im}\left([\psi]^{qp}\right)\mathrm{Re}\left(\left[\gamma_0\right]\left[D_M^\alpha\right]\left[\nabla_\alpha\psi\right]\right)^{qp}+\mathrm{Re}\left([\psi]^{qp}\right)\mathrm{Im}\left(\left[\gamma_0\right]\left[D_M^\alpha\right]\left[\nabla_\alpha\psi\right]\right)^{qp}$

$\frac{\partial\,\mathrm{Im}\left\langle\psi,\sum_\alpha D_M^\alpha[\nabla_\alpha\psi]\right\rangle}{\partial\,\mathrm{Re}\,\psi^{ij}}=\sum_{\alpha pq}\delta_q^i\delta_p^j\,\mathrm{Im}\left(\left[\gamma_0\right]\left[D_M^\alpha\right]\left[\nabla_\alpha\psi\right]\right)^{qp}=\sum_\alpha\mathrm{Im}\left(\left[\gamma_0\right]\left[D_M^\alpha\right]\left[\nabla_\alpha\psi\right]\right)_j^i$

$\frac{\partial\,\mathrm{Im}\left\langle\psi,\sum_\alpha D_M^\alpha[\nabla_\alpha\psi]\right\rangle}{\partial\,\mathrm{Im}\,\psi^{ij}}=\sum_{\alpha pq}-\delta_q^i\delta_p^j\,\mathrm{Re}\left(\left[\gamma_0\right]\left[D_M^\alpha\right]\left[\nabla_\alpha\psi\right]\right)^{qp}=-\sum_\alpha\mathrm{Re}\left(\left[\gamma_0\right]\left[D_M^\alpha\right]\left[\nabla_\alpha\psi\right]\right)_j^i$

$\frac{\partial L_M^\diamondsuit}{\partial\,\mathrm{Re}\,\psi^{ij}}=2a_M\,\mathrm{Im}\left(i\left[\gamma_0\right][\psi]\right)_j^i+\sum_\alpha\mathrm{Im}\left(\left[\gamma_0\right]\left[D_M^\alpha\right]\left[\nabla_\alpha\psi\right]\right)_j^i$

$=\mathrm{Im}\left(2a_Mi\left[\gamma_0\right][\psi]+\sum_\alpha\left[\gamma_0\right]\left[D_M^\alpha\right]\left[\nabla_\alpha\psi\right]\right)_j^i$

$\frac{\partial L_M^\diamondsuit}{\partial\,\mathrm{Im}\,\psi^{ij}}=-2a_M\,\mathrm{Re}\left(i\left[\gamma_0\right][\psi]\right)_j^i-\sum_\alpha\mathrm{Re}\left(\left[\gamma_0\right]\left[D_M^\alpha\right]\left[\nabla_\alpha\psi\right]\right)_j^i$

$=-\mathrm{Re}\left(2a_Mi\left[\gamma_0\right][\psi]+\sum_\alpha\left[\gamma_0\right]\left[D_M^\alpha\right]\left[\nabla_\alpha\psi\right]\right)_j^i$

2) We have already

$\frac{dL_M}{d\,\mathrm{Re}\,\nabla_\alpha\psi^{ij}}=N\,\mathrm{Im}\left([\psi]^*\left[\gamma_0\right]\left[D_M^\alpha\right]\right)_i^j$

$\frac{dL_M}{d\,\mathrm{Im}\,\nabla_\alpha \psi^{ij}} = N\,\mathrm{Re}\left([\psi]^*[\gamma_0][D_M^\alpha]\right)_i^j$

So the equations read:

$0 = N\{\sum_{\alpha k}\mathrm{Im}\left([\psi]^*[\gamma_0][D_M^\alpha]\right)_k^j \mathrm{Re}\,[G_\alpha]_i^k + \mathrm{Re}\left([\psi]^*[\gamma_0][D_M^\alpha]\right)_k^j \mathrm{Im}\,[G_\alpha]_i^k$

$+\,\mathrm{Im}\left([\psi]^*[\gamma_0][D_M^\alpha]\right)_i^k \mathrm{Re}\left[\grave{A}_\alpha\right]_j^k + \mathrm{Re}\left([\psi]^*[\gamma_0][D_M^\alpha]\right)_i^k \mathrm{Im}\left[\grave{A}_\alpha\right]_j^k\}$

$-\sum_\alpha \frac{1}{(\det O')}\frac{d}{d\xi^\alpha}\left(N(\det O')\,\mathrm{Im}\left([\psi]^*[\gamma_0][D_M^\alpha]\right)_i^j\right)$

$+N\,\mathrm{Im}\left(2a_M i[\gamma_0][\psi] + \sum_\alpha [\gamma_0][D_M^\alpha][\nabla_\alpha\psi]\right)_j^i$

$0 = N\{\sum_{\alpha k} -\mathrm{Im}\left([\psi]^*[\gamma_0][D_M^\alpha]\right)_k^j \mathrm{Im}\,[G_\alpha]_i^k + \mathrm{Re}\left([\psi]^*[\gamma_0][D_M^\alpha]\right)_k^j \mathrm{Re}\,[G_\alpha]_i^k$

$-\,\mathrm{Im}\left([\psi]^*[\gamma_0][D_M^\alpha]\right)_i^k \mathrm{Im}\left[\grave{A}_\alpha\right]_j^k + \mathrm{Re}\left([\psi]^*[\gamma_0][D_M^\alpha]\right)_i^k \mathrm{Re}\left[\grave{A}_\alpha\right]_j^k$

$-\sum_\alpha \frac{1}{(\det O')}\frac{d}{d\xi^\alpha}\left(N(\det O')\,\mathrm{Re}\left([\psi]^*[\gamma_0][D_M^\alpha]\right)_i^j\right)$

$-N\,\mathrm{Re}\left(2a_M i[\gamma_0][\psi] + \sum_\alpha [\gamma_0][D_M^\alpha][\nabla_\alpha\psi]\right)_j^i$

$0 = N\sum_\alpha \mathrm{Im}\left([\psi]^*[\gamma_0][D_M^\alpha][G_\alpha] + \left[\grave{A}_\alpha\right]^t[\psi]^*[\gamma_0][D_M^\alpha]\right)_i^j$

$-\sum_\alpha \frac{1}{(\det O')}\frac{d}{d\xi^\alpha}\left(N(\det O')\,\mathrm{Im}\left([\psi]^*[\gamma_0][D_M^\alpha]\right)_i^j\right)$

$+N\,\mathrm{Im}\left(2a_M i[\gamma_0][\psi] + \sum_\alpha [\gamma_0][D_M^\alpha][\nabla_\alpha\psi]\right)_j^i$

$0 = N\sum_{\alpha k} \mathrm{Re}\left([\psi]^*[\gamma_0][D_M^\alpha][G_\alpha] + \left[\grave{A}_\alpha\right]^t[\psi]^*[\gamma_0][D_M^\alpha]\right)_i^j$

$-\sum_\alpha \frac{1}{(\det O')}\frac{d}{d\xi^\alpha}\left(N(\det O')\,\mathrm{Re}\left([\psi]^*[\gamma_0][D_M^\alpha]\right)_i^j\right)$

$-N\,\mathrm{Re}\left(2a_M i[\gamma_0][\psi] + \sum_\alpha [\gamma_0][D_M^\alpha][\nabla_\alpha\psi]\right)_j^i$

3) We have two real equations, that we can combine in the complex matrix equation :

$-N\,\mathrm{Re}\left(2a_M i[\gamma_0][\psi] + \sum_\alpha [\gamma_0][D_M^\alpha][\nabla_\alpha\psi]\right)_j^i$

$+iN\,\mathrm{Im}\left(2a_M i[\gamma_0][\psi] + \sum_\alpha [\gamma_0][D_M^\alpha][\nabla_\alpha\psi]\right)_j^i$

$= -N\left(\left(2a_M i[\gamma_0][\psi] + \sum_\alpha [\gamma_0][D_M^\alpha][\nabla_\alpha\psi]\right)^*\right)_i^j$

$0 = N\sum_\alpha [\psi]^*[\gamma_0][D_M^\alpha][G_\alpha] + \left[\grave{A}_\alpha\right]^t[\psi]^*[\gamma_0][D_M^\alpha]$

$-N\left(2a_M i[\gamma_0][\psi] + \sum_\alpha [\gamma_0][D_M^\alpha][\nabla_\alpha\psi]\right)^*$

$-\sum_\alpha \frac{1}{(\det O')}\frac{d}{d\xi^\alpha}\left(N(\det O')[\psi]^*[\gamma_0][D_M^\alpha]\right)$

By conjugate transpose :

$0 = -2a_M iN[\gamma_0][\psi] - \sum_\alpha \frac{1}{(\det O')}\frac{d}{d\xi^\alpha}\left(N(\det O')[D_M^\alpha]^*[\gamma_0][\psi]\right)$

$+N\sum_\alpha\left([G_\alpha]^*[D_M^\alpha]^*[\gamma_0][\psi] + [D_M^\alpha]^*[\gamma_0][\psi]\overline{\left[\grave{A}_\alpha\right]} - [\gamma_0][D_M^\alpha][\nabla_\alpha\psi]\right)$

$0=-2a_M iN[\gamma_0][\psi]-\sum_\alpha \frac{1}{(\det O')}\frac{d}{d\xi^\alpha}\left(N(\det O')[D_M^\alpha]^*[\gamma_0][\psi]\right)$
$+N\sum_\alpha\left(G_\alpha^a[\kappa_a]^*[D_M^\alpha]^*[\gamma_0][\psi]+\overleftrightarrow{A}_\alpha^a[D_M^\alpha]^*[\gamma_0][\psi]\overline{[\theta_a]}\psi-[\gamma_0][D_M^\alpha][\nabla_\alpha\psi]\right)$
with $\overline{[\theta_a]}=-[\theta_a]^t,[\kappa_a]^*=-[\gamma_0][\kappa_a][\gamma_0]$
$0=-2a_M iN[\gamma_0][\psi]-\sum_\alpha \frac{1}{(\det O')}\frac{d}{d\xi^\alpha}\left(N(\det O')[D_M^\alpha]^*[\gamma_0][\psi]\right)$
$-N\sum_\alpha\left(G_\alpha^a[\gamma_0][\kappa_a][\gamma_0][D_M^\alpha]^*[\gamma_0][\psi]+\overleftrightarrow{A}_\alpha^a[D_M^\alpha]^*[\gamma_0][\psi][\theta_a]^t\psi+[\gamma_0][D_M^\alpha][\nabla_\alpha\psi]\right)$
By left multiplication by $[\gamma_0]$
$0=-2a_M iN[\psi]-\sum_\alpha \frac{1}{(\det O')}\frac{d}{d\xi^\alpha}\left(N(\det O')[\gamma_0][D_M^\alpha]^*[\gamma_0][\psi]\right)$
$-N\sum_\alpha\left(G_\alpha^a[\kappa_a][\gamma_0][D_M^\alpha]^*[\gamma_0][\psi]+\overleftrightarrow{A}_\alpha^a[\gamma_0][D_M^\alpha]^*[\gamma_0][\psi][\theta_a]^t+[D_M^\alpha][\nabla_\alpha\psi]\right)$
$[\gamma_0][D_M^\alpha]^*[\gamma_0]$
$=[\gamma_0]\left(\sum_r a_I O_r^\alpha[\gamma^r]+\sum_\alpha V^\alpha a_D I\right)^*[\gamma_0]$
$=\left(\sum_r a_I O_r^\alpha[\gamma_0][\gamma^r]^*[\gamma_0]+\sum_\alpha V^\alpha a_D I\right)$
$=\left(\sum_r -a_I O_r^\alpha[\gamma^r]+\sum_\alpha V^\alpha a_D I\right)$
$=[D_M'^\alpha]$ with $[\gamma_0][\gamma^r]^*[\gamma_0]=-[\gamma^r]$
That is :

$$\begin{aligned}&\sum_\alpha \frac{d}{d\xi^\alpha}\left(N\det O'[D_M'^\alpha][\psi]\right)\\&=-\mathbf{N}\det\mathbf{O'}\left(2a_M i[\psi]+\sum_{\alpha a}[D_M^\alpha][\nabla_\alpha\psi]+[G_\alpha][D_M'^\alpha][\psi]-[D_M'^\alpha][\psi]\overline{\left[\grave{A}_\alpha\right]}\right)\end{aligned} \tag{102}$$

The scalar product for the state tensor, and consequently the moments, are unchanged by multiplication by a c-number z : $|z|=1$. But as one can see in the equation z must be constant, so we are not fully allowed to normalize ψ.

4) One can expand the derivative. In this equation all variables, but N, are valued at a point $m(f)$. So:
$\sum_\alpha \frac{d}{d\xi^\alpha}\left(N(\det O')[D_M'^\alpha][\psi]\right)$
$=\sum_\alpha \frac{d}{d\xi^\alpha}\left(N(\det O')\left(\sum_r -a_I O_r^\alpha[\gamma^r][\psi]+\sum_\alpha V^\alpha a_D[\psi]\right)\right)$
$=-a_I\sum_{\alpha r}\frac{dN(\det O')O_r^\alpha}{d\xi^\alpha}[\gamma^r][\psi]+a_D\sum_\alpha\frac{dN(\det O')V^\alpha}{d\xi^\alpha}[\psi]+N(\det O')\sum_\alpha[D_M'^\alpha]\frac{d\psi}{d\xi^\alpha}$
and the equation becomes :
$0=-a_I\sum_{\alpha r}\frac{dN(\det O')O_r^\alpha}{d\xi^\alpha}[\gamma^r][\psi]+a_D\sum_\alpha\frac{dN(\det O')V^\alpha}{d\xi^\alpha}[\psi]+N(\det O')\sum_\alpha[D_M'^\alpha]\frac{d\psi}{d\xi^\alpha}$
$+(N\det O')\left(2a_M i[\psi]+\sum_\alpha[D_M^\alpha][\nabla_\alpha\psi]+\left([G_\alpha][D_M'^\alpha][\psi]-[D_M'^\alpha][\psi]\overline{\left[\grave{A}_\alpha\right]}\right)\right)$

$0=-a_I\sum_{\alpha r}\frac{dN(\det O')O_r^\alpha}{d\xi^\alpha}[\gamma^r][\psi]+a_D\sum_\alpha\frac{dN(\det O')V^\alpha}{d\xi^\alpha}[\psi]+2a_Mi(N\det O')[\psi]$
$+(N\det O')\sum_\alpha\left([D_M^\alpha][\nabla_\alpha\psi]+[G_\alpha][D_M'^\alpha][\psi]-[D_M'^\alpha][\psi]\overline{\left[\grave{A}_\alpha\right]}+[D_M'^\alpha]\left[\frac{d\psi}{d\xi^\alpha}\right]\right)$

Let us expand the last term :

$[D_M^\alpha][\nabla_\alpha\psi]+[G_\alpha][D_M'^\alpha][\psi]-[D_M'^\alpha][\psi]\overline{\left[\grave{A}_\alpha\right]}+[D_M'^\alpha]\left[\frac{d\psi}{d\xi^\alpha}\right]$
$=[D_M^\alpha]\left(\left[\frac{d\psi}{d\xi^\alpha}\right]+[G_\alpha][\psi]+[\psi]\left[\grave{A}_\alpha\right]^t\right)+[G_\alpha][D_M'^\alpha][\psi]-[D_M'^\alpha][\psi]\overline{\left[\grave{A}_\alpha\right]}+$
$[D_M'^\alpha]\left[\frac{d\psi}{d\xi^\alpha}\right]$
$=([D_M^\alpha]+[D_M'^\alpha])\left[\frac{d\psi}{d\xi^\alpha}\right]+G_\alpha([D_M^\alpha][\kappa_a]+[\kappa_a][D_M'^\alpha])[\psi]$
$+\grave{A}_\alpha^a[D_M^\alpha][\psi][\theta_a]^t+\overline{\grave{A}_\alpha^a}[D_M'^\alpha][\psi][\theta_a]^t$
$=2\sum_\alpha V^\alpha a_D\left[\frac{d\psi}{d\xi^\alpha}\right]+G_\alpha([D_M^\alpha][\kappa_a]+[\kappa_a][D_M'^\alpha])[\psi]$
$+\left(\grave{A}_\alpha^a[D_M^\alpha]+\overline{\grave{A}_\alpha^a}[D_M'^\alpha]\right)[\psi][\theta_a]^t$
$=2\sum_\alpha V^\alpha a_D\left[\frac{d\psi}{d\xi^\alpha}\right]+G_\alpha\left(a_I\sum_r O_r^\alpha([\gamma^r][\kappa_a]-[\kappa_a][\gamma^r])+2a_DV^\alpha[\kappa_a]\right)[\psi]$
$+\left(a_I\sum_r\left(\grave{A}_\alpha^a-\overline{\grave{A}_\alpha^a}\right)O_r^\alpha[\gamma^r]+\sum_\alpha a_DV^\alpha\left(\grave{A}_\alpha^a+\overline{\grave{A}_\alpha^a}\right)I\right)[\psi][\theta_a]^t$
$=2\sum_\alpha V^\alpha a_D\left[\frac{d\psi}{d\xi^\alpha}\right]+G_\alpha\left(a_I\sum_r O_r^\alpha([\gamma^r][\kappa_a]-[\kappa_a][\gamma^r])+2a_DV^\alpha[\kappa_a]\right)[\psi]$
$+2\left(ia_I\sum_r\left(\operatorname{Im}\grave{A}_\alpha^a\right)O_r^\alpha[\gamma^r]+\sum_\alpha a_DV^\alpha\left(\operatorname{Re}\grave{A}_\alpha^a\right)I\right)[\psi][\theta_a]^t$
$=a_I\sum_r\left(G_r^a([\gamma^r][\kappa_a]-[\kappa_a][\gamma^r])[\psi]+2i\left(\operatorname{Im}\grave{A}_r^a\right)[\gamma^r][\psi][\theta_a]^t\right)$
$+2a_D\sum_\alpha V^\alpha\left(\left[\frac{d\psi}{d\xi^\alpha}\right]+[G_\alpha][\psi]+\left(\operatorname{Re}\grave{A}_\alpha^a\right)[\psi][\theta_a]^t\right)$

The expanded equation reads:

$0=-a_I\sum_{\alpha r}\frac{dN(\det O')O_r^\alpha}{d\xi^\alpha}[\gamma^r][\psi]+a_D\sum_\alpha\frac{dN(\det O')V^\alpha}{d\xi^\alpha}[\psi]+2a_Mi(N\det O')[\psi]$
$+(N\det O')a_I\sum_r\left(G_r^a([\gamma^r][\kappa_a]-[\kappa_a][\gamma^r])[\psi]+2i\left(\operatorname{Im}\grave{A}_r^a\right)[\gamma^r][\psi][\theta_a]^t\right)$
$+2a_D\sum_\alpha V^\alpha\left(\left[\frac{d\psi}{d\xi^\alpha}\right]+[G_\alpha][\psi]+\left(\operatorname{Re}\grave{A}_\alpha^a\right)[\psi][\theta_a]^t\right)$
$0=2a_Mi(N\det O')[\psi]-\sum_\alpha\frac{dN(\det O')O_r^\alpha}{d\xi^\alpha}[\gamma^r][\psi])$
$+a_I\sum_r((N\det O')\left(G_r^a([\gamma^r][\kappa_a]-[\kappa_a][\gamma^r])[\psi]+2i\left(\operatorname{Im}\grave{A}_r^a\right)[\gamma^r][\psi][\theta_a]^t\right)$
$+a_D\sum_\alpha\left(\frac{dN(\det O')V^\alpha}{d\xi^\alpha}[\psi]+2(N\det O')V^\alpha\left(\left[\frac{d\psi}{d\xi^\alpha}\right]+[G_\alpha][\psi]+\left(\operatorname{Re}\grave{A}_\alpha^a\right)[\psi][\theta_a]^t\right)\right)$

It will be most useful latter.

4) One can compute the equation with respect to ψ_R,ψ_L

With : $[\nabla_\alpha \psi] = \begin{bmatrix} \partial_\alpha \psi_R + \frac{1}{2}\sum_1^3 \left(G_\alpha^{a+3} - iG_\alpha^a\right)\sigma_a \psi_R + \psi_R \left[\grave{A}_\alpha\right]^t \\ \partial_\alpha \psi_L - \frac{1}{2}\sum_1^3 \left(G_\alpha^{a+3} + iG_\alpha^a\right)\sigma_a \psi_L + \psi_L \left[\grave{A}_\alpha\right]^t \end{bmatrix}$
the result is the two matrix equations, which are not simple...

$$\frac{1}{N\det O'}\sum_\alpha \frac{d}{d\xi^\alpha}\left(\left(-a_D V^\alpha \psi_R + a_I \sum_r O_r^\alpha \sigma_r \psi_L\right)(N\det O')\right) \tag{103}$$
$$= 2ia_M\psi_R + a_D\sum_\alpha V^\alpha \left(\partial_\alpha \psi_R + \sum_{a=1}^3 \left(G_\alpha^{a+3} - iG_\alpha^a\right)\sigma_a\psi_R + 2\psi_R\left[\operatorname{Re}\grave{A}_\alpha\right]^t\right)$$
$$+a_I\sum_r \sigma_r\left(\sum_\alpha O_r^\alpha \partial_\alpha\psi_L - \sum_{a=1}^3\left(\delta_r^a\left(G_r^{a+3} - iG_r^a\right)\sigma_r + iG_r^a\sigma_a\right)\psi_L + 2i\psi_L\left[\operatorname{Im}\grave{A}_\alpha\right]^t\right)$$

$$\frac{1}{N\det O'}\sum_\alpha \frac{d}{d\xi^\alpha}\left(\left(a_I \sum_r O_r^\alpha \eta^{rr}\sigma_r \psi_R - a_D V^\alpha \psi_L\right)(N\det O')\right) \tag{104}$$
$$= a_I\sum_r \eta^{rr}\sigma_r\left(\partial_\alpha\psi_R + \sum_{a=1}^3\left(\delta_r^a\left(G_r^{a+3} + iG_r^a\right)\sigma_r - iG_r^a\sigma_a\right)\psi_R + 2i\psi_R\left[\operatorname{Im}\grave{A}_\alpha\right]^t\right)$$
$$+2ia_M\psi_L + a_D\sum_\alpha V^\alpha\left(\partial_\alpha\psi_L - \sum_{a=1}^3\left(G_\alpha^{a+3} + iG_\alpha^a\right)\sigma_a\psi_L + 2\psi_L\left[\operatorname{Re}\grave{A}_\alpha\right]^t\right)$$

Usually in this kind of equations the Dirac operator brings trouble. Indeed it exchanges the two subspaces F^+, F^-, so one cannot have $\psi_R \equiv 0$ or $\psi_L \equiv 0$ without $\psi = 0$. It does not happen here thanks to the introduction of V.

15.4.2 Moments

From these equation we can deduce new equations for the moments. We will use the expanded equation :

$0 = 2a_M i\,(N\det O')\,[\psi]$
$+a_I\sum_r((N\det O')\left(G_r^a\left([\gamma^r][\kappa_a] - [\kappa_a][\gamma^r]\right)[\psi] + 2i\left(\operatorname{Im}\grave{A}_r^a\right)[\gamma^r][\psi][\theta_a]^t\right) - \sum_\alpha \frac{dN(\det O')O_r^\alpha}{d\xi^\alpha}[\gamma^r][\psi])$
$+a_D\sum_\alpha\left(\frac{dN(\det O')V^\alpha}{d\xi^\alpha}[\psi] + 2\,(N\det O')\,V^\alpha\left(\left[\frac{d\psi}{d\xi^\alpha}\right] + [G_\alpha][\psi] + \left(\operatorname{Re}\grave{A}_\alpha^a\right)[\psi][\theta_a]^t\right)\right)$

1) Taking the scalar product on the left with ψ :

$0 = 2a_M i \left(N \det O'\right) \langle \psi, \psi \rangle - a_I \sum_\alpha \frac{dN(\det O')O_r^\alpha}{d\xi^\alpha} \langle \psi, [\gamma^r] [\psi] \rangle$

$+a_D \sum_\alpha \frac{dN(\det O')V^\alpha}{d\xi^\alpha} \langle \psi, \psi \rangle +$

$+a_I \left(N \det O'\right) \sum_r \left(G_r^a \langle \psi, ([\gamma^r] [\kappa_a] - [\kappa_a] [\gamma^r]) [\psi] \rangle + 2i \left(\operatorname{Im} \grave{A}_r^a \right) \left\langle \psi, [\gamma^r] [\psi] [\theta_a]^t \right\rangle \right)$

$+a_D 2 \left(N \det O'\right) \sum_\alpha V^\alpha \left(\left\langle \psi, \left[\frac{d\psi}{d\xi^\alpha} \right] \right\rangle + \langle \psi, [G_\alpha] [\psi] \rangle + \left(\operatorname{Re} \grave{A}_\alpha^a \right) \left\langle \psi, [\psi] [\theta_a]^t \right\rangle \right)$

It comes:

$0 = \left(2a_M i \left(N \det O'\right) + a_D \sum_\alpha \frac{dN(\det O')V^\alpha}{d\xi^\alpha} \right) \langle \psi, \psi \rangle$

$+a_D \sum_\alpha \left(2 \left(N \det O'\right) V^\alpha \left(\left\langle \psi, \frac{d\psi}{d\xi^\alpha} \right\rangle + G_\alpha^a i P_a + \left(\operatorname{Re} \grave{A}_\alpha^a \right) i\rho_a \right) \right)$

$+a_I \sum_r \left(N \det O'\right) \sum_a (\begin{pmatrix} G_r^a \left(\langle \psi, [\gamma^r] [\kappa_a] [\psi] \rangle - \langle \psi, [\kappa_a] [\gamma^r] [\psi] \rangle \right) \\ -2i \left(\operatorname{Im} \grave{A}_r^a \right) [\mu_A]_a^r \end{pmatrix}$

$+2i \sum_\alpha \frac{dN(\det O')O_r^\alpha}{d\xi^\alpha} J_r)$

with :

$\left\langle \psi, [\psi] [\theta_a]^t \right\rangle = i\rho_a$

$\left\langle \psi, \gamma^r [\psi] [\theta_a]^t \right\rangle = - [\mu_A]_a^r$

$\langle \psi, [\kappa_a] [\psi] \rangle = iP_a$

$\langle \psi, \gamma^r \psi \rangle = -2iJ_r$

2) We are left with : $\langle \psi, [\gamma^r] [\kappa_a] [\psi] \rangle - \langle \psi, [\kappa_a] [\gamma^r] [\psi] \rangle$

But : $\langle \psi, [\kappa_a] [\gamma^r] [\psi] \rangle = Tr \left([\psi]^* [\gamma_0] [\kappa_a] [\gamma^r] [\psi] \right)$

$\overline{Tr \left([\psi]^* [\gamma_0] [\kappa_a] [\gamma^r] [\psi] \right)} = Tr \left(\left([\psi]^* [\gamma_0] [\kappa_a] [\gamma^r] [\psi] \right)^* \right)$

$= Tr \left([\psi]^* [\gamma^r]^* [\kappa_a]^* [\gamma_0] [\psi] \right) = Tr \left([\psi]^* [\gamma_0] [\gamma^r] [\gamma_0] [\gamma_0] [\kappa_a] [\gamma_0] [\gamma_0] [\psi] \right)$

$= Tr \left([\psi]^* [\gamma_0] [\gamma^r] [\kappa_a] [\psi] \right) = \langle \psi, [\gamma^r] [\kappa_a] [\psi] \rangle$

with $[\gamma_0] [\gamma^r]^* [\gamma_0] = - [\gamma^r] , [\kappa_a]^* = - [\gamma_0] [\kappa_a] [\gamma_0]$

So $\langle \psi, [\kappa_a] [\gamma^r] [\psi] \rangle = \overline{\langle \psi, [\gamma^r] [\kappa_a] [\psi] \rangle}$

$\langle \psi, [\gamma^r] [\kappa_a] [\psi] \rangle - \langle \psi, [\kappa_a] [\gamma^r] [\psi] \rangle$

$= \langle \psi, [\gamma^r] [\kappa_a] [\psi] \rangle - \overline{\langle \psi, [\gamma^r] [\kappa_a] [\psi] \rangle}$

$= 2i \operatorname{Im} \langle \psi, [\gamma^r] [\kappa_a] [\psi] \rangle = 2i \left([J] [\widetilde{\kappa}_a] \right)_r$

with : $\operatorname{Im} \langle \psi, [\gamma^r] [\kappa_a] \psi \rangle = \left([J] [\widetilde{\kappa}_a] \right)_r$

and : $\left\langle \psi, \frac{d\psi}{d\xi^\alpha} \right\rangle$

But : $\frac{d}{d\xi^\alpha} \langle \psi, \psi \rangle = \left\langle \frac{d\psi}{d\xi^\alpha}, \psi \right\rangle + \left\langle \psi, \frac{d\psi}{d\xi^\alpha} \right\rangle = 2 \operatorname{Re} \left\langle \psi, \frac{d\psi}{d\xi^\alpha} \right\rangle$

$\left\langle \psi, \frac{d\psi}{d\xi^\alpha} \right\rangle = \frac{1}{2} \frac{d}{d\xi^\alpha} \langle \psi, \psi \rangle + i \operatorname{Im} \left\langle \psi, \frac{d\psi}{d\xi^\alpha} \right\rangle$

The equation reads:

$0 = \left(2a_M i \left(N \det O'\right) + a_D \sum_\alpha \frac{dN(\det O')V^\alpha}{d\xi^\alpha}\right) \langle\psi,\psi\rangle$
$+a_D \sum_\alpha \left(2\left(N \det O'\right) V^\alpha \left(\frac{1}{2}\frac{d}{d\xi^\alpha}\langle\psi,\psi\rangle + i \operatorname{Im}\left\langle\psi, \frac{d\psi}{d\xi^\alpha}\right\rangle + G^a_\alpha i P_a + \left(\operatorname{Re} \grave{A}^a_\alpha\right) i\rho_a\right)\right)$
$+2ia_I \sum_r \left(\left(N \det O'\right) \sum_a \left(G^a_r \left([J]\left[\widetilde{\kappa}_a\right]\right)_r - \left(\operatorname{Im} \grave{A}^a_r\right)\left[\mu_R - \mu_L\right]^r_a\right) + \sum_\alpha \frac{dN(\det O')O^\alpha_r}{d\xi^\alpha} J_r\right)$

3) Taking the real part :
$0 = a_D \sum_\alpha \frac{dN(\det O')V^\alpha}{d\xi^\alpha}\langle\psi,\psi\rangle + a_D\left(N \det O'\right)\sum_\alpha V^\alpha \frac{d}{d\xi^\alpha}\langle\psi,\psi\rangle$

$$\sum_\alpha \frac{d}{d\xi^\alpha}\left(\left(N \det O'\right) V^\alpha \langle\psi,\psi\rangle\right) = \mathbf{0} \tag{105}$$

Remind the convention about the creation and annihiliation of particles : $\psi(m) = 0$ means no particle . So $\left(N \det O'\right)\langle\psi,\psi\rangle$ can be seen roughly as a density of particles, this equation expresses a conservation law of the flow of moving particles.

4) Taking the imaginary part :
$0 = 2a_M\left(N \det O'\right)\langle\psi,\psi\rangle + 2\left(N \det O'\right) a_D \sum_\alpha V^\alpha \left(\operatorname{Im}\left\langle\psi, \frac{d\psi}{d\xi^\alpha}\right\rangle + G^a_\alpha P_a + \left(\operatorname{Re} \grave{A}^a_\alpha\right)\rho_a\right)$
$+2a_I \sum_r \left(\left(N \det O'\right) \sum_a \left(G^a_r \left([J]\left[\widetilde{\kappa}_a\right]\right)_r - \left(\operatorname{Im} \grave{A}^a_r\right)\left[\mu_A\right]^r_a\right) + \sum_\alpha \frac{dN(\det O')O^\alpha_r}{d\xi^\alpha} J_r\right)$
$0 = a_M \langle\psi,\psi\rangle + a_D \sum_\alpha V^\alpha \left(\operatorname{Im}\left\langle\psi, \frac{d\psi}{d\xi^\alpha}\right\rangle + \sum_a \left(G^a_\alpha P_a + \rho_a \operatorname{Re} \grave{A}^a_\alpha\right)\right) +$
$a_I \left(\sum_{\alpha r} \frac{dN(\det O')O^\alpha_r}{N(\det O')d\xi^\alpha} J_r + \left([J]\left[G_r\right]\right)_r - \sum_a \left(\operatorname{Im} \grave{A}^a_r\right)\left[\mu_A\right]^r_a\right)$
With $\sum_\alpha V^\alpha \left(\operatorname{Im}\left\langle\psi, \frac{d\psi}{d\xi^\alpha}\right\rangle\right) = \operatorname{Im}\left\langle\psi, \sum_\alpha V^\alpha \frac{d\psi}{d\xi^\alpha}\right\rangle = \operatorname{Im}\left\langle\psi, \frac{d\psi}{d\xi^0}\right\rangle$

$$\begin{aligned} 0 &= a_M \langle\psi,\psi\rangle + a_D \operatorname{Im}\left\langle\psi, \frac{d\psi}{d\xi^0}\right\rangle + a_D \sum_{\alpha a} V^\alpha \left(G^a_\alpha P_a + \rho_a \operatorname{Re} \grave{A}^a_\alpha\right) \\ &\quad + a_I \left(\sum_{\alpha r} \frac{dN(\det O')O^\alpha_r}{N(\det O')d\xi^\alpha} J_r + \left([J]\left[G_r\right]\right)_r - \sum_a \left(\operatorname{Im} \grave{A}^a_r\right)\left[\mu_A\right]^r_a\right)\end{aligned} \tag{106}$$

As we have :
$L_M = a_M \langle\psi,\psi\rangle + a_I \left(-\sum_{\alpha,r} O^\alpha_r \frac{dJ_r}{d\xi^\alpha} + \left([J]\left[G_r\right]\right)_r - \sum_a \left(\operatorname{Im} \grave{A}^a_r\right)\left[\mu_A\right]^r_a\right)$
$+a_D \sum_\alpha V^\alpha \left(\operatorname{Im}\left\langle\psi, \frac{d\psi}{d\xi^\alpha}\right\rangle + \sum_a \left(G^a_\alpha P_a + \rho_a \operatorname{Re} \grave{A}^a_\alpha\right)\right)$
$0 = L_M + a_I \left(\sum_{\alpha r} \frac{dN(\det O')O^\alpha_r}{N(\det O')d\xi^\alpha} J_r + \left([J]\left[G_r\right]\right)_r - \sum_a \left(\operatorname{Im} \grave{A}^a_r\right)\left[\mu_A\right]^r_a\right)$

$-a_I \left(-\sum_{\alpha,r} O_r^\alpha \frac{dJ_r}{d\xi^\alpha} + ([J]\,[G_r])_r - \sum_a \left(\operatorname{Im} \grave{A}_r^a \right) [\mu_A]_a^r \right)$
$0 = L_M + a_I \sum_{\alpha r} \left(\frac{dN(\det O')O_r^\alpha}{N(\det O')d\xi^\alpha} J_r + O_r^\alpha \frac{dJ_r}{d\xi^\alpha} \right)$

$$\mathbf{0} = \mathbf{N} (\det O')\, \mathbf{L}_M + \mathbf{a}_I \sum_{\alpha r} \frac{d}{d\xi^\alpha} \left(N (\det O')\, O_r^\alpha J_r \right) \tag{107}$$

$N (\det O')\, L_M$ can be seen roughly as a density of the energy of the particles, and $N (\det O')\, O_r^\alpha J_r$ as the "internal energy" , so this equation can be seen as a conservation of energy. Notice that there is no clear equivalent of rest mass (but for J_0 ?).

5) Remind the equation for the scalar curvature :
$R = -\frac{N}{a_G} \left(2a_M \langle \psi, \psi \rangle + \frac{3}{2} a_I \operatorname{Im} \langle \psi, D\psi \rangle + 2a_D \sum_\alpha \operatorname{Im} \langle \psi, V^\alpha \nabla_\alpha \rangle \right)$
It reads :
$a_G R (\det O') = -2N L_M (\det O') + \frac{1}{2} N (\det O')\, a_I \operatorname{Im} \langle \psi, D\psi \rangle$
$= 2a_I \sum_{\alpha r} \frac{dN(\det O')O_r^\alpha J_r}{d\xi^\alpha} + \frac{1}{2} N (\det O')\, a_I \operatorname{Im} \langle \psi, D\psi \rangle$

$$\mathbf{a}_G \mathbf{R} (\det O') = \mathbf{a}_I \left(2 \sum_{\alpha r} \frac{dN (\det O')\, O_r^\alpha J_r}{d\xi^\alpha} + \frac{1}{2} N (\det O') \operatorname{Im} \langle \psi, D\psi \rangle \right) \tag{108}$$

The scalar curvature is entirely linked to the kinematic part of the particles : are involved neither the field (but they are involved in the covariant derivative) or the velocity.

15.5 Trajectory

The equation for the trajectory of particles reads :
$\forall \alpha : a_D \sum_\beta V^\beta \partial_\beta \left(\operatorname{Im} \langle \psi, \nabla_\alpha \psi \rangle \det O' \right)$
$= L_M \partial_\alpha \det O' + (\det O')\, a_D \sum_{a\beta} \left(V^\beta \left(\partial_\alpha G_\beta^a \right) P_a + V^\beta \rho_a \left(\partial_\alpha \operatorname{Re} \grave{A}_\beta^a \right) \right)$
$+ (\det O')\, a_I \sum_{\beta r} \left(-\frac{dJ_r}{d\xi^\alpha} \left(\partial_\alpha O_r^\beta \right) + \sum_a \left(\partial_\alpha G_r^a \right) ([J]\,[\widetilde{\kappa}_a])_r - [\mu_A]_a^r \operatorname{Im} \left(\partial_\alpha \grave{A}_r^\alpha \right) \right)$
$L_M \partial_\alpha \det O'$ is obtained through one previous equation :
$0 = N (\det O')\, L_M + a_I \sum_{\alpha r} \frac{dN(\det O')O_r^\alpha J_r}{d\xi^\alpha}$
So the equation links the derivative $\partial_\beta \left(\operatorname{Im} \langle \psi, \nabla_\alpha \psi \rangle \det O' \right)$ to the moments and the velocity V (the derivatives of V figures in the state equation).

1) The equation 56 reads :

$\forall\alpha:0=\frac{d}{d\xi^0}\left(\frac{\partial\mathcal{L}_M}{\partial V^\alpha}\right)$

$+2\sum_{a,\beta\gamma}\{\left(\partial_\alpha\operatorname{Re}\grave{A}^a_\beta\right)\partial_\gamma\left(\frac{\partial\mathcal{L}_F}{\partial\operatorname{Re}\mathcal{F}^a_{A,\beta\gamma}}\right)+\left(\partial_\alpha\operatorname{Im}\grave{A}^a_\beta\right)\partial_\gamma\left(\frac{\partial\mathcal{L}_F}{\partial\operatorname{Im}\mathcal{F}^b_{A,\beta\gamma}}\right)$

$+\frac{\partial\mathcal{L}_F}{\partial\operatorname{Re}\mathcal{F}^a_{A,\beta\gamma}}\operatorname{Re}\left[\partial_\alpha\grave{A}_\beta,\grave{A}_\gamma\right]^a+\frac{\partial\mathcal{L}_F}{\partial\operatorname{Im}\mathcal{F}^a_{A,\beta\gamma}}\operatorname{Im}\left[\partial_\alpha\grave{A}_\beta,\grave{A}_\gamma\right]^a\}$

$+\sum_{a\beta}\left(\frac{\partial\mathcal{L}_F}{\partial G^a_\alpha}\partial_\alpha G^a_\beta+2\sum_\gamma\left(\left(\partial_\alpha G^a_\beta\right)\partial_\gamma\left(\frac{\partial\mathcal{L}_F}{\partial\mathcal{F}^a_{G\beta\gamma}}\right)+\frac{\partial\mathcal{L}_F}{\partial\mathcal{F}^a_{G\beta\gamma}}\left[\partial_\alpha G_\beta,G_\gamma\right]^a\right)\right)$

$+\sum_{i\beta}\left(\frac{dL_F(\det O')}{dO'^i_\beta}-\sum_\gamma\frac{d}{d\xi^\gamma}\left(\frac{dL_F(\det O')}{d\partial_\gamma O'^i_\beta}\right)\right)\partial_\alpha O'^i_\beta$

We have the derivatives (all derivatives in L_F do not involve f) :

$\frac{\partial\mathcal{L}_F}{\partial\mathcal{F}^a_{G\beta\gamma}}=a_G\left(O^\gamma_{p_a}O^\beta_{q_a}-O^\gamma_{q_a}O^\beta_{p_a}\right)\det O'$

$\frac{dL_F}{d\operatorname{Re}\mathcal{F}^a_{A,\beta\gamma}}=a_F\operatorname{Re}\mathcal{F}^{a,\beta\gamma}_A;\frac{dL_F}{d\operatorname{Im}\mathcal{F}^a_{A,\beta\gamma}}=a_F\operatorname{Im}\mathcal{F}^{a,\beta\gamma}_A$

$\frac{dL}{d\partial_\alpha O'^i_\gamma}=0$

$\frac{dNL_M}{dV^\alpha}=a_DN\operatorname{Im}\langle\psi,\nabla_\alpha\psi\rangle$

$\frac{dL_F}{dO'^i_\beta}=-2\sum_\gamma O^\gamma_i\left(a_F\sum_\mu\operatorname{Re}\left(\mathcal{F}^{\beta\mu}_A,\mathcal{F}_{A\gamma\mu}\right)+a_G\sum_{a\mu}\mathcal{F}^a_{G\gamma\mu}\left(O^\mu_{p_a}O^\beta_{q_a}-O^\mu_{q_a}O^\beta_{p_a}\right)\right)$

$\frac{dL_F(\det O')}{dO'^i_\beta}=\frac{dL_F}{dO'^i_\beta}(\det O')+L_F\frac{d\det O'}{dO'^i_\beta}$

$=-2(\det O')\sum_\gamma O^\gamma_i\left(a_F\sum_\mu\operatorname{Re}\left(\mathcal{F}^{\beta\mu}_A,\mathcal{F}_{A\gamma\mu}\right)+a_G\sum_{a\mu}\mathcal{F}^a_{G\gamma\mu}\left(O^\mu_{p_a}O^\beta_{q_a}-O^\mu_{q_a}O^\beta_{p_a}\right)\right)+$
$L_FO^\beta_i(\det O')$

with $\frac{\partial(\det O')}{\partial O'^i_\beta}=O^\beta_i(\det O')$

The equation reads :

$\forall\alpha:0=\frac{d}{d\xi^0}\left(a_DN(\det O')\operatorname{Im}\langle\psi,\nabla_\alpha\psi\rangle\right)$

$+2a_F\sum_{a,\beta\gamma}\{\left(\partial_\alpha\operatorname{Re}\grave{A}^a_\beta\right)\partial_\gamma\left(\operatorname{Re}\mathcal{F}^{a,\beta\gamma}_A\det O'\right)+\left(\partial_\alpha\operatorname{Im}\grave{A}^a_\beta\right)\partial_\gamma\left(\operatorname{Im}\mathcal{F}^{a,\beta\gamma}_A\det O'\right)$

$+\operatorname{Re}\mathcal{F}^{a,\beta\gamma}_A\operatorname{Re}\left[\partial_\alpha\grave{A}_\beta,\grave{A}_\gamma\right]^a\det O'+\operatorname{Im}\mathcal{F}^{a,\beta\gamma}_A\operatorname{Im}\left[\partial_\alpha\grave{A}_\beta,\grave{A}_\gamma\right]^a\det O'\}$

$+2a_G\sum_{a\beta\gamma}\{\partial_\alpha G^a_\beta\partial_\gamma\left(\left(O^\gamma_{p_a}O^\beta_{q_a}-O^\gamma_{q_a}O^\beta_{p_a}\right)\det O'\right)$
$+\left(O^\gamma_{p_a}O^\beta_{q_a}-O^\gamma_{q_a}O^\beta_{p_a}\right)\det O'\left[\partial_\alpha G_\beta,G_\gamma\right]^a\}$

$+\sum_{i\beta}\{-2(\det O')\sum_\gamma O^\gamma_i\left(a_F\sum_\mu\operatorname{Re}\left(\mathcal{F}^{\beta\mu}_A,\mathcal{F}_{A\gamma\mu}\right)+a_G\sum_{a\mu}\mathcal{F}^a_{G\gamma\mu}\left(O^\mu_{p_a}O^\beta_{q_a}-O^\mu_{q_a}O^\beta_{p_a}\right)\right)$
$+L_FO^\beta_i(\det O')\}\partial_\alpha O'^i_\beta$

$\forall\alpha:0=\frac{d}{d\xi^0}\left(a_DN(\det O')\operatorname{Im}\langle\psi,\nabla_\alpha\psi\rangle\right)$

$+L_F\partial_\alpha\det O'+2a_F\sum_{a,\beta\gamma}\operatorname{Re}\left(\overline{\left(\partial_\alpha\grave{A}^a_\beta\right)}\partial_\gamma\left(\mathcal{F}^{a,\beta\gamma}_A\det O'\right)+\mathcal{F}^{a,\beta\gamma}_A\overline{\left[\partial_\alpha\grave{A}_\beta,\grave{A}_\gamma\right]^a}\det O'\right)$

$+2a_G\sum_{a\beta\gamma}\{\partial_\alpha G^a_\beta\partial_\gamma\left(\left(O^\gamma_{p_a}O^\beta_{q_a}-O^\gamma_{q_a}O^\beta_{p_a}\right)\det O'\right)$
$+\left(O^\gamma_{p_a}O^\beta_{q_a}-O^\gamma_{q_a}O^\beta_{p_a}\right)\det O'\left[\partial_\alpha G_\beta,G_\gamma\right]^a\}$

$-2(\det O')\sum_{i\beta\gamma\mu} O_i^\gamma \left(\partial_\alpha O_\beta'^i\right)\left(a_F \operatorname{Re}\left(\mathcal{F}_A^{\beta\mu}, \mathcal{F}_{A\gamma\mu}\right) + a_G \mathcal{F}_{G\gamma\mu}^a \left(O_{p_a}^\mu O_{q_a}^\beta - O_{q_a}^\mu O_{p_a}^\beta\right)\right)$

2) Let us compute the third term :
$2a_F \sum_{\beta\gamma} \operatorname{Re}\left(\left(\partial_\alpha \grave{A}_\beta, \partial_\gamma\left(\mathcal{F}_A^{a,\beta\gamma}\det O'\right)\right) + \left(\left[\partial_\alpha \grave{A}_\beta, \grave{A}_\gamma\right], \mathcal{F}_A^{\beta\gamma}\det O'\right)\right)$
Using the identity in T_1U^c :
$\forall \overrightarrow{\theta}, \overrightarrow{\theta}_1, \overrightarrow{\theta}_2 \in T_1U^c : \left(\left[\overrightarrow{\theta}, \overrightarrow{\theta}_1\right], \overrightarrow{\theta}_2\right) = -\left(\overrightarrow{\theta}_1, \left[\overrightarrow{\theta}, \overrightarrow{\theta}_2\right]\right)$
$\left(\left[\partial_\alpha \grave{A}_\beta, \grave{A}_\gamma\right], \mathcal{F}_A^{\beta\gamma}\det O'\right)$
$= -\left(\left[\grave{A}_\gamma, \partial_\alpha \grave{A}_\beta,\right], \mathcal{F}_A^{\beta\gamma}\det O'\right)$
$= \left(\partial_\alpha \grave{A}_\beta, \left[\grave{A}_\gamma, \mathcal{F}_A^{\beta\gamma}\det O'\right]\right)$
$2a_F \sum_{\beta\gamma}\left(\left(\partial_\alpha \grave{A}_\beta, \partial_\gamma\left(\mathcal{F}_A^{a,\beta\gamma}\det O'\right)\right) + \left(\left[\partial_\alpha \grave{A}_\beta, \grave{A}_\gamma\right], \mathcal{F}_A^{\beta\gamma}\det O'\right)\right)$
$= 2a_F \sum_{\beta\gamma} \operatorname{Re}\left(\partial_\alpha \grave{A}_\beta, \left[\grave{A}_\gamma, \mathcal{F}_A^{\beta\gamma}\det O'\right] + \partial_\gamma\left(\mathcal{F}_A^{a,\beta\gamma}\det O'\right)\right)$
$= 2a_F \sum_{\beta\gamma} \operatorname{Re}\left(\left[\grave{A}_\gamma, \mathcal{F}_A^{\beta\gamma}\det O'\right] + \partial_\gamma\left(\mathcal{F}_A^{a,\beta\gamma}\det O'\right), \partial_\alpha \grave{A}_\beta\right)$
$= 2a_F \sum_{\beta\gamma}\sum_a \operatorname{Re}\left(\overline{\left(\left[\grave{A}_\gamma, \mathcal{F}_A^{\beta\gamma}\det O'\right] + \partial_\gamma\left(\mathcal{F}_A^{a,\beta\gamma}\det O'\right)\right)}^a \left(\partial_\alpha \grave{A}_\beta^a\right)\right)$
with equation 94 on shell :
$N\left(a_D V^\beta \rho_a - i a_I \sum_r O_r^\beta \left[\mu_A\right]_a^r\right)\det O'$
$= -2a_F \sum_\gamma \left(\left[\grave{A}_\beta, \mathcal{F}_A^{\beta\gamma}\det O'\right]^a + \partial_\gamma\left(\mathcal{F}_A^{a\beta\gamma}(\det O')\right)\right)$
So :
$2a_F \sum_{\beta\gamma}\left(\left(\partial_\alpha \grave{A}_\beta, \partial_\gamma\left(\mathcal{F}_A^{a,\beta\gamma}\det O'\right)\right) + \left(\left[\partial_\alpha \grave{A}_\beta, \grave{A}_\gamma\right], \mathcal{F}_A^{\beta\gamma}\det O'\right)\right)$
$= -N(\det O')\sum_{a\beta} \operatorname{Re}\left(\overline{\left(a_D V^\beta \rho_a - i a_I \sum_r O_r^\beta \left[\mu_A\right]_a^r\right)}\left(\partial_\alpha \grave{A}_\beta^a\right)\right)$
$= -N(\det O')\sum_{a\beta} \operatorname{Re}\left(\left(a_D V^\beta \rho_a + i a_I \sum_r O_r^\beta \left[\mu_A\right]_a^r\right)\left(\partial_\alpha \grave{A}_\beta^a\right)\right)$
$= -N(\det O')\left(a_D \sum_{a\beta} V^\beta \rho_a \operatorname{Re}\left(\partial_\alpha \grave{A}_\beta^a\right) - a_I \sum_r O_r^\beta \left[\mu_A\right]_a^r \operatorname{Im}\left(\partial_\alpha \grave{A}_\beta^a\right)\right)$

3) The fourth term :

$$2a_G \sum_{a\beta\gamma}\begin{pmatrix} \left(\partial_\alpha G_\beta^a\right)\partial_\gamma\left(\left(O_{p_a}^\gamma O_{q_a}^\beta - O_{q_a}^\gamma O_{p_a}^\beta\right)\det O'\right) \\ + \left(O_{p_a}^\gamma O_{q_a}^\beta - O_{q_a}^\gamma O_{p_a}^\beta\right)\det O'\left[\partial_\alpha G_\beta, G_\gamma\right]^a \end{pmatrix}$$

a) We have seen (equation 87) that :
$\sum_{\beta=0}^3 \partial_\beta\left(\left(O_{p_a}^\beta O_{q_a}^\alpha - O_{q_a}^\beta O_{p_a}^\alpha\right)\det O'\right) = \left(\sum_r O_r^\alpha c_a^r + \left(O_{q_a}^\alpha D_{p_a} - O_{p_a}^\alpha D_{q_a}\right)\right)(\det O')$
So :

$\sum_{a\beta}\left(\partial_\alpha G^a_\beta\right)\left(\sum_\gamma \partial_\gamma\left(\left(O^\gamma_{p_a}O^\beta_{q_a}-O^\gamma_{q_a}O^\beta_{p_a}\right)\det O'\right)\right)$
$=\sum_{a\beta r}\left(\partial_\alpha G^a_\beta\right)\left(O^\beta_r c^r_a+\left(O^\beta_{q_a}D_{p_a}-O^\beta_{p_a}D_{q_a}\right)\right)\left(\det O'\right)$

b) $\sum_{a\beta\gamma}\left(O^\gamma_{p_a}O^\beta_{q_a}-O^\gamma_{q_a}O^\beta_{p_a}\right)\left[\partial_\alpha G_\beta, G_\gamma\right]^a=\sum_{\beta\gamma b}\left(O^\gamma_{p_b}O^\beta_{q_b}-O^\gamma_{q_b}O^\beta_{p_b}\right)\left[\partial_\alpha G_\beta, G_\gamma\right]^b$
$=\sum_{ab\beta\gamma r}O^\beta_r\left(\partial_\alpha G^a_\beta\right)\left(O^\gamma_{p_b}\delta^r_{q_b}-O^\gamma_{q_b}\delta^r_{p_b}\right)\left[\overrightarrow{\kappa}_a, G_\gamma\right]^b$
$=\sum_{a\beta r}O^\beta_r\left(\partial_\alpha G^a_\beta\right)\sum_b\left(O^\gamma_{p_b}\delta^r_{q_b}-O^\gamma_{q_b}\delta^r_{p_b}\right)\left[\overrightarrow{\kappa}_a, G_\gamma\right]^b$
$\sum_b\left(O^\gamma_{p_b}\delta^r_{q_b}-O^\gamma_{q_b}\delta^r_{p_b}\right)\left[\overrightarrow{\kappa}_a, G_\gamma\right]^b$
$=\sum_b\delta^r_{q_b}\left[\overrightarrow{\kappa}_a, G_{p_b}\right]^b-\delta^r_{p_b}\left[\overrightarrow{\kappa}_a, G_{q_b}\right]^b$
$=\sum_{bc=1}^6 G^b_{ac}\left(\delta^r_{q_b}G^c_{p_b}-\delta^r_{p_b}G^c_{q_b}\right)$
We have seen (equation 84) that :
$[T_G]^{ar}=\sum_{bc=1}^6 G^b_{ac}\left(\delta^r_{q_b}G^c_{p_b}-\delta^r_{p_b}G^c_{q_b}\right)$
So : $\sum_{a\beta\gamma}\left(O^\gamma_{p_a}O^\beta_{q_a}-O^\gamma_{q_a}O^\beta_{p_a}\right)\left[\partial_\alpha G_\beta, G_\gamma\right]^a=\sum_{a\beta r}O^\beta_r\left(\partial_\alpha G^a_\beta\right)[T_G]^{ar}$

c) $2a_G\sum_{a\beta\gamma}\{\left(\partial_\alpha G^a_\beta\right)\partial_\gamma\left(\left(O^\gamma_{p_a}O^\beta_{q_a}-O^\gamma_{q_a}O^\beta_{p_a}\right)\det O'\right)$
$+\left(O^\gamma_{p_a}O^\beta_{q_a}-O^\gamma_{q_a}O^\beta_{p_a}\right)\det O'\left[\partial_\alpha G_\beta, G_\gamma\right]^a\}$
$=2a_G\left(\sum_{a\beta r}\left(\partial_\alpha G^a_\beta\right)\left(O^\beta_r c^r_a+\left(O^\beta_{q_a}D_{p_a}-O^\beta_{p_a}D_{q_a}\right)\right)+O^\beta_r\left(\partial_\alpha G^a_\beta\right)[T_G]^{ar}\right)\det O'$
$=2a_G\left(\sum_{a\beta r}\left(\partial_\alpha G^a_\beta\right)O^\beta_r\left(c^r_a+\delta^r_{q_a}D_{p_a}-\delta^r_{p_a}D_{q_a}+[T_G]^{ar}\right)\right)\det O'$
But from the equation 87:
$[T_G]^{ar}=-c^r_a+\delta^r_{p_a}D_{q_a}-\delta^r_{q_a}D_{p_a}-\frac{N}{2a_G}\left(a_I\left([J]\left[\widetilde{\kappa}_a\right]\right)_r+a_D V^r P_a\right)$
$2a_G\sum_{a\beta\gamma}\left(\left(\partial_\alpha G^a_\beta\right)\partial_\gamma\left(\left(O^\gamma_{p_a}O^\beta_{q_a}-O^\gamma_{q_a}O^\beta_{p_a}\right)\det O'\right)+\left(O^\gamma_{p_a}O^\beta_{q_a}-O^\gamma_{q_a}O^\beta_{p_a}\right)\det O'\left[\partial_\alpha G_\beta, G_\gamma\right]^a\right)$
$=2a_G\left(\sum_{a\beta r}\left(\partial_\alpha G^a_\beta\right)O^\beta_r\left(-\frac{N}{2a_G}\left(a_I\left([J]\left[\widetilde{\kappa}_a\right]\right)_r+a_D V^r P_a\right)\right)\right)\det O'$
$=-N\sum_{a\beta r}\left(\partial_\alpha G^a_\beta\right)O^\beta_r\left(a_I\left([J]\left[\widetilde{\kappa}_a\right]\right)_r+a_D V^r P_a\right)\det O'$

4) The last term :
$2\sum_{i\beta\gamma\mu}O^\gamma_i\left(\partial_\alpha O'^i_\beta\right)\left(a_F\operatorname{Re}\left(\mathcal{F}^{\beta\mu}_A, \mathcal{F}_{A\gamma\mu}\right)+a_G\mathcal{F}^a_{G\gamma\mu}\left(O^\mu_{p_a}O^\beta_{q_a}-O^\mu_{q_a}O^\beta_{p_a}\right)\right)\left(\det O'\right)$
From equation 97 we have :
$\forall\alpha,\beta:\delta^\alpha_\beta L$
$=a_I N\operatorname{Im}\langle\psi,\gamma^\alpha\nabla_\beta\psi\rangle+2a_F\sum_\gamma\operatorname{Re}\left(\mathcal{F}^{\alpha\gamma}_A, \mathcal{F}_{A\beta\gamma}\right)+2a_G\sum_{a\lambda}\mathcal{F}^a_{G\beta\gamma}\left(O^\gamma_{p_a}O^\alpha_{q_a}-O^\gamma_{q_a}O^\alpha_{p_a}\right)$
$2\left(a_F\sum_\gamma\operatorname{Re}\left(\mathcal{F}^{\alpha\gamma}_A, \mathcal{F}_{A\beta\gamma}\right)+a_G\sum_{a\lambda}\mathcal{F}^a_{G\beta\gamma}\left(O^\gamma_{p_a}O^\alpha_{q_a}-O^\gamma_{q_a}O^\alpha_{p_a}\right)\right)$
$=\left(\delta^\alpha_\beta L-a_I N\operatorname{Im}\langle\psi,\gamma^\alpha\nabla_\beta\psi\rangle\right)$
So :
$2\sum_{i\beta\gamma\mu}O^\gamma_i\left(\partial_\alpha O'^i_\beta\right)\left(a_F\operatorname{Re}\left(\mathcal{F}^{\beta\mu}_A, \mathcal{F}_{A\gamma\mu}\right)+a_G\mathcal{F}^a_{G\gamma\mu}\left(O^\mu_{p_a}O^\beta_{q_a}-O^\mu_{q_a}O^\beta_{p_a}\right)\right)\left(\det O'\right)$

$= \sum_{i\beta\gamma} O_i^\gamma \left(\partial_\alpha O'^i_\beta\right) \left(\left(\delta^\beta_\gamma L - a_I N \operatorname{Im}\left\langle \psi, \gamma^\beta \nabla_\gamma \psi \right\rangle\right)\right) (\det O')$
$= (\partial_\alpha \det O') L + a_I N (\det O') \sum_{i\beta\gamma} O_i^\gamma O'^i_\beta \left(\partial_\alpha O^\beta_r\right) \operatorname{Im} \langle \psi, \gamma^r \nabla_\gamma \psi \rangle$
$= (\partial_\alpha \det O') L + a_I N (\det O') \sum_\beta \left(\partial_\alpha O^\beta_r\right) \operatorname{Im} \langle \psi, \gamma^r \nabla_\beta \psi \rangle$

5) The equation becomes :
$\forall \alpha : \frac{d}{d\xi^0} \left(a_D N (\det O') \operatorname{Im} \langle \psi, \nabla_\alpha \psi \rangle\right) = N a_D \det O' \sum_{a\beta} V^\beta \left(\left(\partial_\alpha G^a_\beta\right) P_a + \rho_a \left(\partial_\alpha \operatorname{Re} \grave{A}^a_\beta\right)\right) +$
$+ a_I \det O' N \sum_r \{ \left(\partial_\alpha G^a_\beta\right) O^\beta_r ([J][\widetilde{\kappa}_a])_r - O^\beta_r [\mu_A]^r_a \left(\partial_\alpha \operatorname{Im} \grave{A}^a_\beta\right)$
$- O_i^\gamma \left(\partial_\alpha O'^i_\beta\right) \operatorname{Im} \left\langle \psi, \gamma^\beta \nabla_\gamma \psi \right\rangle + N L_M \partial_\alpha \det O' \}$
That is:
$\frac{d}{d\xi^0} \left(a_D N (\det O') \operatorname{Im} \langle \psi, \nabla_\alpha \psi \rangle\right)$
$= N L_M \partial_\alpha \det O' + (N \det O') a_D \sum_{a\beta} V^\beta \left(\left(\partial_\alpha G^a_\beta\right) P_a + \rho_a \left(\partial_\alpha \operatorname{Re} \grave{A}^a_\beta\right)\right) +$
$+ N \det O' a_I \sum_{\beta r} \left(\partial_\alpha O^\beta_r \operatorname{Im} \langle \psi, \gamma^r \nabla_\beta \psi \rangle + \sum_a O^\beta_r \left(\partial_\alpha G^a_\beta ([J][\widetilde{\kappa}_a])_r - [\mu_A]^r_a \partial_\alpha \operatorname{Im} \grave{A}^a_\beta\right)\right)$
We have seen that :
$\langle \psi, \gamma^r \nabla_\beta \psi \rangle = -\frac{dJ_r}{d\xi^\beta} + \sum_a G^a_\beta ([J][\widetilde{\kappa}_a])_r - [\mu_A]^r_a \operatorname{Im} \grave{A}^\alpha_\beta$
$\sum_{\beta r} \left(\left(\partial_\alpha O^\beta_r\right) \operatorname{Im} \langle \psi, \gamma^r \nabla_\beta \psi \rangle + \sum_a O^\beta_r \left(\left(\partial_\alpha G^a_\beta\right) ([J][\widetilde{\kappa}_a])_r - [\mu_A - \mu_L]^r_a \left(\partial_\alpha \operatorname{Im} \grave{A}^a_\beta\right)\right)\right)$
$= \sum_{\beta r} \{ \left(\partial_\alpha O^\beta_r\right) \left(-\frac{dJ_r}{d\xi^\beta} + \sum_a G^a_\beta ([J][\widetilde{\kappa}_a])_r - [\mu_R - \mu_L]^r_a \operatorname{Im} \grave{A}^\alpha_\beta\right)$
$+ \sum_a O^\beta_r \left(\left(\partial_\alpha G^a_\beta\right) ([J][\widetilde{\kappa}_a])_r - [\mu_A - \mu_L]^r_a \left(\partial_\alpha \operatorname{Im} \grave{A}^a_\beta\right)\right) \}$
$= \sum_{\beta r} \left(\left(-\frac{dJ_r}{d\xi^\beta} \left(\partial_\alpha O^\beta_r\right) + \sum_a \left(\partial_\alpha O^\beta_r G^a_\beta\right) ([J][\widetilde{\kappa}_a])_r - [\mu_A]^r_a \operatorname{Im} \left(\partial_\alpha O^\beta_r \grave{A}^\alpha_\beta\right)\right)\right)$
$= \sum_{\beta r} \left(\left(-\frac{dJ_r}{d\xi^\beta} \left(\partial_\alpha O^\beta_r\right) + \sum_a \left(\partial_\alpha G^a_r\right) ([J][\widetilde{\kappa}_a])_r - [\mu_A]^r_a \operatorname{Im} \left(\partial_\alpha \grave{A}^\alpha_r\right)\right)\right)$
$\frac{d}{d\xi^0} \left(a_D N (\det O') \operatorname{Im} \langle \psi, \nabla_\alpha \psi \rangle\right)$
$= N L_M \partial_\alpha \det O' + (N \det O') a_D \sum_{a\beta} V^\beta \left(\left(\partial_\alpha G^a_\beta\right) P_a + \rho_a \left(\partial_\alpha \operatorname{Re} \grave{A}^a_\beta\right)\right) +$
$+ (N \det O') a_I \sum_{\beta r} \left(\left(-\frac{dJ_r}{d\xi^\beta} \left(\partial_\alpha O^\beta_r\right) + \sum_a \left(\partial_\alpha G^a_r\right) ([J][\widetilde{\kappa}_a])_r - [\mu_A]^r_a \operatorname{Im} \left(\partial_\alpha \grave{A}^\alpha_r\right)\right)\right)$

6) The derivative reads:
$\frac{d}{d\xi^0} \left(a_D (\det O') N \operatorname{Im} \langle \psi, \nabla_\alpha \psi \rangle\right)$
$= a_D N \frac{d}{d\xi^0} \left((\det O') \operatorname{Im} \langle \psi, \nabla_\alpha \psi \rangle\right)$ as N is constant along ξ^0
$a_D N \frac{d}{d\xi^0} \left((\det O') \operatorname{Im} \langle \psi, \nabla_\alpha \psi \rangle\right) = a_D N \sum_\beta \frac{df^\beta}{d\xi^0} \partial_\beta \left(\operatorname{Im} \langle \psi, \nabla_\alpha \psi \rangle \det O'\right)$
$= a_D N \sum_\beta V^\beta \partial_\beta \left(\operatorname{Im} \langle \psi, \nabla_\alpha \psi \rangle \det O'\right)$

so, assuming that N$\neq$ 0 we have the equation :

$$\begin{aligned}
&\forall \alpha : a_D \sum_\beta V^\beta \partial_\beta \left(\operatorname{Im}\langle \psi, \nabla_\alpha \psi\rangle \det O'\right) \\
&= L_M \partial_\alpha \det O' + (\det O')\, a_D \sum_{a\beta} \left(V^\beta \left(\partial_\alpha G^a_\beta\right) P_a + V^\beta \rho_a \left(\partial_\alpha \operatorname{Re} \grave{A}^a_\beta\right)\right) \\
&+ (\det O')\, a_I \sum_{\beta r} \left(-\frac{dJ_r}{d\xi^\alpha}\left(\partial_\alpha O^\beta_r\right) + \sum_a \left(\partial_\alpha G^a_r\right)\left([J]\left[\widetilde{\kappa}_a\right]\right)_r - \left[\mu_A\right]^r_a \operatorname{Im}\left(\partial_\alpha \grave{A}^\alpha_r\right)\right)
\end{aligned} \tag{109}$$

16 CHOOSING A GAUGE

We have, all together 16m+36 first degree partial differential equations for 16m+36 variables. But the 16 tetrad coefficients O' are defined within a SO(3,1) matrix and so we could fix 10 parameters. Now we intend to use this gauge freedom, by choosing a chart and a tetrad.

1) The most physical choice for the chart is just that we have built in the beginning :

- the time vector n is taken as the 0 basis vector, both for the tetrad and the chart. In this section $(t, \xi = (\xi^1, \xi^2, \xi^3))$ are the coordinates along n on one hand, and in a chart of S(t) on the other hand. So we have

$$O^\alpha_0 = \delta^\alpha_0, O'^i_0 = \delta^i_0 \tag{110}$$

$\frac{d}{dt} O^\alpha_i (\xi, t) = n^\beta \partial_\beta O (\xi, t) = \partial_4 O^\alpha_i (\xi, t) = \partial_t O^\alpha_i (\xi, t)$

- the vectors ∂^i are parallel transported along a gravitational geodesic :

$\forall i : \widehat{\nabla} \partial_i (n) = 0 = G^a_0 \left[\widetilde{\kappa}_a\right]^j_i \partial_j = G^a_0 \left(\left(\delta^{jp} \eta_{iq} - \delta^{jq} \eta_{ip}\right)\right) \partial_j = G^a_0 \eta_{iq} \partial_p - B^a_0 \eta_{ip} \partial_q = 0$

$G^a_0 \eta_{pp} \partial_q = 0; G^a_0 \partial_p = 0$

$\Rightarrow$

$$\forall a : G^a_0 = 0 \tag{111}$$

- the world lines of the particles are therefore such that $f^0 = t \Rightarrow \frac{df^0}{d\xi^\alpha} = \delta^0_\alpha$
- the function N does not depend on t

2) As we have seen the structure coefficients are key variables in the model :

$c_{pq}^r = \sum_{\alpha\beta} O_\alpha'^r \left(O_p^\beta \partial_\beta O_q^\alpha - O_q^\beta \partial_\beta O_p^\alpha \right) = [\partial_p, \partial_q]^r$

There are 24 coefficients denoted $c_a^r = c_{p_a q_a}^r$ which are not independant. Indeed there are the Jacobi identities coming from the commutator rules :

$\forall (a,b,c) : [\partial_a, [\partial_b, \partial_c]] + [\partial_b, [\partial_c, \partial_a]] + [\partial_c, [\partial_a, \partial_b]] = 0$

$\Rightarrow \forall (a,b,c), d = 0..3 : \sum_{i=1}^4 \left(c_{bc}^i c_{ai}^d + c_{ca}^i c_{bi}^d + c_{ab}^i c_{ci}^d \right) = 0$

With the present assumptions :

$c_{pq}^0 = \sum_{\alpha\beta} \left(O_p^\beta \partial_\beta O_q^0 - O_q^\beta \partial_\beta O_p^0 \right) = 0, c_{0q}^r = \sum_\alpha O_\alpha'^r \partial_t O_q^\alpha,$

and we have 18 non null coefficients linked by 12 identities (with d=1,2,3), so only 6 of them can be considered as independant.

The Jacobi identities can be conveniently put on the matrix form:

$$\begin{bmatrix} (c_3^2 - c_2^3) & (c_3^1 + c_1^3) & (c_2^1 - c_1^2) & 0 & 0 & 0 \\ (c_6^3 + c_5^2) & c_5^1 & -c_6^1 & -c_1^1 & -c_1^2 & -c_1^3 \\ c_4^2 & -(c_4^1 + c_6^3) & c_6^2 & c_2^1 & c_2^2 & c_2^3 \\ -c_4^3 & c_5^3 & c_4^1 + c_5^2 & -c_3^1 & -c_3^2 & -c_3^3 \end{bmatrix} \begin{bmatrix} c_1^1 & c_1^2 & c_1^3 \\ c_2^1 & c_2^2 & c_2^3 \\ c_3^1 & c_3^2 & c_3^3 \\ c_4^1 & c_4^2 & c_4^3 \\ c_5^1 & c_5^2 & c_5^3 \\ c_6^1 & c_6^2 & c_6^3 \end{bmatrix} = 0$$

The divergence D of the tetrad takes here the following value :

$D_1 = (c_3^2 - c_2^3)$

$D_2 = (-c_3^1 + c_1^3)$

$D_3 = (c_2^1 - c_1^2)$

$D_0 = -(c_4^1 + c_5^2 + c_6^3)$

The gravitational Noether current Y_G is conserved, and it is directly related to the structure coefficients. Thus it should be doable to take these coefficients as constant in a first approximation.

3) From $G_0^a = 0$ we deduce :

$-K_1^0 + K_5^3 - K_6^2 = c_1^4 - c_5^3 + c_6^2$

$-K_2^0 + K_6^1 - K_4^3 = -c_6^1 + c_2^0 + c_4^3$

$-K_3^0 - K_5^1 + K_4^2 = c_5^1 - c_4^2 + c_3^0$

$K_4^0 + K_2^3 - K_3^2 = -2c_4^0$

$K_5^0 + K_3 - K_1^3 = -2c_5^0$

$K_6^0 - K_2 + K_1^2 = -2c_6^0$

In most of the problems where the coefficients structure are either assumed constants, or linked by some symmetry, it is convenient to keep K as

the variables, and the K_a^0 are given by the previous relations. The gravitational potential becomes :

$$2G_r^a =$$
$$\begin{bmatrix} r=1 \\ K_1^1 + c_1^1 - K_2^2 - c_2^2 - K_3^3 - c_3^3 \\ 2K_1^2 + 2c_2^1 + 2c_6^0 \\ 2K_1^3 + 2c_3^1 - 2c_5^0 \\ -K_4^1 - 2c_4^1 + K_5^2 + K_6^3 \\ -2K_4^2 - 2c_4^2 \\ -2K_4^3 - 2c_4^3 \end{bmatrix}
\begin{bmatrix} r=2 \\ 2K_2^1 + 2c_1^2 - 2c_6^0 \\ -K_1^1 - c_1^1 + K_2^2 + c_2^2 - K_3^3 - c_3^3 \\ 2K_2^3 + 2c_3^2 + 2c_4^0 \\ -2K_5^1 - 2c_5^1 \\ K_4^1 - K_5^2 - 2c_5^2 + K_6^3 \\ -2K_5^3 - 2c_5^3 \end{bmatrix}
\begin{bmatrix} r=3 \\ 2K_3^1 + 2c_1^3 + 2c_5^0 \\ +2K_3^2 + 2c_2^3 - 2c_4^0 \\ -K_1^1 - c_1^1 - K_2^2 - c_2^2 + K_3^3 + c_3^3 \\ -2K_6^1 - 2c_6^1 \\ -2K_6^2 - 2c_5^3 \\ K_4^1 + K_5^2 - K_6^3 - 2c_6^3 \end{bmatrix}$$

4) In the Special Relativity picture all the structure coefficient are null, but the connection **G** , depending on V and J, is not null. It can be seen as the stress-tensor of the system. But of course a lagrangian with the scalar curvature R is questionable in this picture, at the least. However the fields equations 94 are still fully valid.

Part V
SYMMETRIES

Symmetries are everywhere in physics, but the same word is used with many different meanings. They arouse some of the most difficult questions in physics, so they cannot be dealt with lightly. According to the relativity principle physical reality does not depend on the mathematics that we used, and the measures of two observers for the same system should be *equivariant*, meaning that they can be deduced from each other with only the knowledge of the mathematical rules to pass from one observer to the other. We have used abondantly this principle in this paper. If these measures are *identical* we shall conclude that the system itself is special : there is a physical symmetry. In both cases the mathematics involved are similar, they rely on group theory, and physical symmetries are identified by a departure from the general rule of equivariance, which must be set up first. But their physical meaning is very different. Equivariance is a consistency requirement, assuming that the right mathematical structure has been picked up to describe a set of measures. A physical symmetry should be an experimental outcome, requiring changes in a pre-existent model by adding assumptions about the configuration of the system or the mechanisms that it tries to describe.

It can happen that all observers get the same measures for a physical quantity : it can be a special case of equivariance, requiring that the quantity should be described by scalar functions (in differential geometry), or that the configuration of the system is isotropic. But most often physical symmetries can be seen by a specific class of observers only and it is convenient to characterize the symmetry by this class. For instance a cylinder looks the same for the observers located in the same plane orthogonal to its axis, and one concludes that the body has a "cylindrical symmetry". This is convenient, and authorized by the usual duality active / passive measures, but to some extent only, as we will see later.

Symmetries are often a specific characteristic of the system itself : there are not an issue, just a good mean to alleviate the computations. In theoretical physics, and in the kind of model that is involved here, one looks for symmetries that occur in any system, whatever its initial configuration, possibly for certain kinds of physical objets, as a way to classify these objects. They have been extensively studied in particles physics, where one discerns

3 "symmetry modes" (Guidry [7]), that we will address in several ways :

- the "Wigner mode" : the lagrangian is symmetric and the vacuum is invariant

- the "Goldstone mode" : there is a global symmetry for the lagrangian but the vacuum is not invariant

- the "Higgs mode" : there is a local symmetry for the lagrangian, and the vacuum is not invariant

The "vacuum" is essentially a quantum concept. In our picture one can see the vacuum as the value of the section $\psi : M \to E_M$, without the map f, which defines the initial state of the system. This section has a life on its own, closely linked to the physical objects involved. To say that the vacuum is symmetric is a strong assumption about the true nature of the particles and the fields. Notably the choice of the group to represent ψ depends on how we see the particles. One has a "Wigner symmetry" if there is a subgroup H of $Spin(3,1) \times U$ such that 2 states ψ, ψ' related by a gauge in H look identical. One has this kind of symmetry whenever one uses an hermitian scalar product : if $\psi' = z\psi$ with z a c-number of module 1 one gets the same value for the lagrangian. This symmetry is usually seen as a mathematical artefact that the physicists discard by a normalization of the state vectors.

The change of orientation of space-time (CPT invariance) or of the signature of the metric are related to the "Goldstone mode".

We will address also with more details the spatial and physical symmetries, which are of particular importance.

We will also give a general picture of the "symmetry breakdown", which is a fundamental mechanism in particles physics to give a mass to the bosons.

17 CPT INVARIANCE

In the first section we noticed that, if the universe is orientable, a change of gauge that does not preserve the orientation cannot be acceptable. And it is easy to check in the model that the equations are significantly altered in a change of space (P) or time (T) orientation. But a global change of both the space and the time orientation could be acceptable, because it does keep the orientation of the 4 dimensional universe. Such a change is indeed a change of chart, that we have studied in the "covariance" section. It is easy to check that the equations are not altered, and it is just the consequence of their covariance. Notice that the Green function N keeps its sign (it is positive on

S(0)).

But there is a significant change with the Noether currents : the moments J, P,ρ,μ are scalar functions, invariant by a change of chart or gauge. And in the model they act by multiplication with vectors (usually ∂_r or V^α) . So if the orientation of this vector is reversed, one should change the sign of J, P,ρ,μ to stay consistent. This operation, denoted as "C symmetry" (for "charge"), is not included in the "gauge transform" package, it must be dealt with separately.

The usual interpretation is that the label (positive, negative,..) that we give to particles is purely conventional and linked with the choice made to name "direct" or "indirect" the geometric basis. The interesting point is that the gravitational charges are equally affected.

18 SIGNATURE

1) If the signature becomes (- + + +) -> (+ - - -) : η^{rr} takes the new values $\eta^{00} = +1, r > 0 : \eta^{rr} = -1$. We have seen that the Clifford algebras Cl(1,3) and Cl(3,1) are not isomorphic, but the representation (F, ρ) of Cl(4,C) is not affected, so the γ matrices and the split of F are unchanged. The change of the signature impacts only the quantities defined through the map Υ, that is :

- the matrices $[\kappa_a] : a < 4 : [\kappa'_a] = -[\kappa_a]$
- the matrices $[\gamma^r] : [\gamma'^r]^* = -i\eta^{rr} [\gamma^r]^*$

The scalar product is unchanged.

One goes from one signature to the other by :

$\psi \to \psi' = \sum_{jk} -i\eta^{jj}\psi^{jk} e_j \otimes f_k \Leftrightarrow [\psi'] = -i [\eta] [\psi]$

2) In the choice of a lagrangian the key argument was the imaginary and real parts of $\sum_r Tr ([\psi^*] \gamma_0\gamma^r [\partial_\alpha\psi])$. With the other signature it becomes

$\sum_r Tr ([\psi'^*] \gamma_0\gamma'^r [\partial_\alpha\psi'])$

$= \sum_r Tr (i [\psi]^* [\eta] [\gamma_0] (i\eta^{rr}) [\gamma^r] (-i [\eta] [\partial_\alpha\psi]))$

$= \sum_r i\eta^{rr} Tr ([\psi]^* ([\eta] \gamma_0 [\gamma^r] [\eta]) [\partial_\alpha\psi])$

So we would be lead to take the opposite choice for the lagrangian.

3) Therefore check the consequences of the choice of signature is not a simple thing...Intuitively all the equations which can be expressed with the moments should not be impacted, but for a change of definition for the

momenta. This leaves the equation of state which depends heavily of the coordinates. I will let this issue open. If ever this model was pertinent, this change would, perhaps, be a mean to test the physical significance of the signature. After all we have so far no sensible explanation for the imbalance between matter and anti-matter.

19 SPATIAL SYMMETRIES

Spatial symmetry is a subtle matter. We will first define what could be a spatial symmetry. In the general framework used to describe the state of particles we will characterize "symmetric states". It will be done without any model, even without the principle of least action. Eventually, with the simple model built previously we will try to bring some light to the results.

19.1 Definition

1) We will say that there is a spatial symmetry if the measures done by a class of observers, using different spatial gauges, are identical for any system, at least for some kinds of particles. Mathematically there is :

- a set of state tensors, described as the product $F_S \otimes W_S$, of two vector subspaces $W_S \subset W$,$F_S \subset F$,

- a subgroup S of of Spin(3,1),

such that two observers (located a the same point as the particle) whose frames (∂_r) differ by $s \in S$ get the same measures on the system whenever $\psi \in F_S \otimes W_S$

2) Let us assume that there is a symmetry on the Wigner mode. So there is a group H and an action $\varkappa : H \to L(F \otimes W; F \otimes W)$ such that the states ψ and $\widehat{\psi} = \varkappa(h)\psi, \forall h \in H$ are imperceptible.

The spatial symmetry is then fully defined by the condition :

$\forall \psi \in F_S \otimes W_S, \forall s \in S, \forall u \in U, \exists h \in H :$

$\vartheta(s,u)\psi = \varkappa(h)\vartheta(1,u)\psi \Leftrightarrow \forall \psi \in F_S \otimes W_S, \vartheta(S,U)\psi \subset \varkappa(H)\psi$

which implies

$\forall \psi \in F_S \otimes W_S, \forall (s,u) \in S \times U : \vartheta(s,u)\psi \in F_S \otimes W_S$

because $\vartheta(s',u')\vartheta(s,u)\psi = \vartheta(s's,u'u)\psi = \varkappa(h'h)\vartheta(1,u'u)\psi$

Therefore $(F_S \otimes W_S, \vartheta)$ is a linear representation of SxU.

3) We will focus on the important case where H=U(1) : two states ψ, ψ' are imperceptible if there is a c-complex number such that $\psi' = z\psi, |z| = 1$. It happens whenever the lagrangian is defined from hermitian scalar products. So the condition reads :

$\forall \psi \in F_S \otimes W_S, \forall s \in S, \forall u \in U, \exists \theta \in \mathbb{R} : \vartheta(s,u)\psi = e^{i\theta}\vartheta(1,u)\psi$

θ can depend on s, u and ψ.

Notice that this definition, per se, is quite general, and does not involve the principle of least action.

19.2 The space of spacially symmetric states

1) Let be the m vectors of F such that :

$\phi_j = \sum_{i=1}^{4} \psi^{ij} e_i$ so $\psi = \sum_{j=1}^{m} \phi_j \otimes f_j$

$\vartheta(s,u)\psi = \sum_{k,l} \psi^{kl} [\rho \circ \Upsilon(s)]_k^i [\chi(u)]_l^j e_i \otimes f_j$

So if $\psi \in F_S \otimes W_S : \forall s \in S, \forall u \in U$:

$\sum_{j=1}^{m} \vartheta(s,u)(\phi_j \otimes f_j) = \sum_{j=1}^{m} e^{i\theta(s)}\phi_j \otimes f_j = \sum_{j=1}^{m} (\rho \circ \Upsilon(s)\phi_j) \otimes (\chi(u) f_j)$

Let u=1 : $\sum_{j=1}^{m} (\rho \circ \Upsilon(s)\phi_j) \otimes f_j = \sum_{j=1}^{m} \left(e^{i\theta(s)}\phi_j\right) \otimes f_j$

$\Rightarrow \forall \psi \in F_S \otimes W_S, \forall s \in S : \rho \circ \Upsilon(s)\phi_j = e^{i\theta(s)}\phi_j$

So for each j $(\phi_j, \rho \circ \Upsilon|_S)$ is a linear representation of S, of complex dimension 1. Besides the scalar matrices the only subgroups S of Spin(3,1) which admit such a representation are the abelian groups generated by $(\overrightarrow{\kappa}_a)_{a=1}^{3}$ on one hand, $(\overrightarrow{\kappa}_a)_{a=4}^{6}$ on the other.

In the standard representation of SO(3,1) the first is the subgroup of rotations with fixed axis of components $(0, y_1, y_2, y_3)$ in an orthonormal basis with $y_1^2 + y_2^2 + y_3^2 = 1$, generated by the matrices $\jmath \circ \mu\left(\exp \tau\left(y_1 \overrightarrow{\kappa}_1 + y_2 \overrightarrow{\kappa}_2 + y_3 \overrightarrow{\kappa}_3\right)\right)$.

The second is the subgroup of space-time rotations of matrices

$\jmath \circ \mu\left(\exp \tau\left(z_1 \overrightarrow{\kappa}_4 + z_2 \overrightarrow{\kappa}_5 + z_3 \overrightarrow{\kappa}_6\right)\right)$, with $z_1^2 + z_2^2 + z_3^2 = 1$.

They correspond to the coordinate change between two observers moving at the spatial speed $(z_1 c \tanh \tau, z_2 c \tanh \tau, z_3 c \tanh \tau)$.

2) The action of these two subgroups can be described by $\rho \circ \Upsilon\left(\sum r^a \overrightarrow{\kappa}_a\right)$ where the components r^a are fixed and the index a runs from 1 to 3 for the first and 4 to 6 for the second subgroup.

So :

$\forall \tau \in \mathbb{R} : \vartheta\left(\exp \tau \sum r^a \overrightarrow{\kappa}_a, 1_U\right)\psi$

$= \sum_{i,k,j} \psi^{kj} \left[\rho \circ \Upsilon\left(\exp \tau \sum r^a \overrightarrow{\kappa}_a\right)\right]_k^i e_i \otimes f_j$

$= \sum_j \left[\rho \circ \Upsilon\left(\exp \tau \sum r^a \overrightarrow{\kappa}_a\right)\right](\phi_j) \otimes f_j$

$\vartheta\left(\exp \tau \sum r^{a} \overrightarrow{\kappa}_{a}, 1_{G}\right) \psi=\sum_{i, j} e^{i \theta(\tau)} \psi^{i j} e_{i} \otimes f_{j}=\sum_{j} e^{i \theta(\tau)} \phi_{j} \otimes f_{j}$
$\Leftrightarrow \forall_{j}:\left[\rho \circ \Upsilon\left(\exp \tau \sum r^{a} \overrightarrow{\kappa}_{a}\right)\right]\left(\phi_{j}\right)=e^{i \theta(\tau)} \phi_{j}$

By differenciating with respect to τ in $\tau=0$:

$(\rho \circ \Upsilon)^{\prime}(1)\left(\sum r^{a} \overrightarrow{\kappa}_{a}\right) \phi_{j}=i \theta^{\prime}(0) \phi_{j}=\sum_{a} r^{a}\left[\kappa_{a}\right] \phi_{j}=\left[\kappa_{S}\right] \phi_{j}$

So a necessary condition is that the vectors ϕ_{j} are eigen vectors of the matrices $\left[\kappa_{S}\right]=\sum r^{a}\left[\kappa_{a}\right]$ with the same imaginary eigen value. Conversely $(\rho \circ \Upsilon, F)$ is a representation of Spin(3,1), so $\left(\rho^{\prime} \circ \Upsilon^{\prime}(1), F\right)$ is a representation of o(3,1), and the matrices of the representation of Spin(3,1) in F are the exponential of the matrices of the representation of o(3,1), which are here the matrices $\left[\kappa_{S}\right]$. Thus : $\rho \circ \Upsilon\left(\exp \tau \sum r_{a}^{a} \overrightarrow{\kappa}\right)=\exp \left(\left[\kappa_{S}\right]\right)$ .If ϕ_{j} is eigen vector of $\left[\kappa_{S}\right]$ with the eigen value $i \lambda, \lambda \in \mathbb{R}$ it is an eigen vector of $\exp \tau\left[\kappa_{S}\right]$ with eigen value $\exp \tau i \lambda$ and this condition is also sufficient.

3) The matrices $\left[\kappa_{a}\right]$ are built from the matrices γ, which are defined within conjugation and so their eigen values do not depend of the choice of the γ matrices and we can take those defined previously. The eigen values are $\pm \frac{1}{2} i$ for the first subgroup and $\pm \frac{1}{2}$ for the second. Therefore the spatial rotations are the only possible case, and $\theta(s)=\frac{\epsilon}{2}, \epsilon=\pm 1$. The $\left[\kappa_{S}\right]$ matrices are normal, so diagonalizable and each of their eigen spaces $F_{\epsilon}(r)$ is 2 dimensional.

The axis of the rotation is fixed by the eigen vector r (which is space like) and the direction of the rotation by ϵ.

4) Let $\left(u_{\epsilon}(r), v_{\epsilon}(r)\right)$ be a basis of $F_{\epsilon}(r)$ .For each symmetric state ψ there is some r such that:

$\psi=\sum_{j=1}^{m} \phi_{j} \otimes f_{j}, \phi_{j} \in F_{\epsilon}(r)$

that we can write : $\phi_{j}=a_{j} u_{\epsilon}(r)+b_{j} v_{\epsilon}(r),\left(a_{j}, b_{j}\right) \in \mathbb{C}$

So the set of symmetric states is the subspace of $F \otimes W$:

$\psi=u_{\epsilon}(r) \otimes\left(\sum_{j=1}^{m} a_{j} f_{j}\right)+v_{\epsilon}(r) \otimes\left(\sum_{j=1}^{m} b_{j} f_{j}\right)=u_{\epsilon}(r) \otimes \sigma_{1}+v_{\epsilon}(r) \otimes \sigma_{2}$
with some $\sigma_{1}, \sigma_{2} \in W$

5) If $\phi \in F_{\epsilon}(r): \phi=\gamma_{+} \phi+\gamma_{-} \phi$

$\sum_{a=1}^{3} r^{a}\left[\kappa_{a}\right]\left(\gamma_{\epsilon} \phi\right)=\gamma_{\epsilon} \sum_{a=1}^{3} r^{a}\left[\kappa_{a}\right] \phi=\epsilon \frac{1}{2} i \gamma_{\epsilon} \phi$ with $\gamma_{\epsilon}\left[\kappa_{a}\right]=\left[\kappa_{a}\right] \gamma_{\epsilon}$

thus $\gamma_{+} \phi$ and $\gamma_{-} \phi$ are still eigen vectors with the same eigen value, and $F_{\epsilon}(r)$ split between F^{+}, F^{-} and the states can be written :

$$\psi=\phi_{+} \otimes \sigma_{+}+\phi_{-} \otimes \sigma_{-} \tag{112}$$

$\Rightarrow [\psi] = [\phi_+] [\sigma_+]^t + [\phi_-] [\sigma_-]^t$ with $\gamma_+\phi_+ = \phi_+; \gamma_-\phi_- = \phi_-$

$$\sum_{a=1}^{3} r^a [\kappa_a] \phi_+ = \epsilon\frac{1}{2}i\phi_+; \sum_{a=1}^{3} r^a [\kappa_a] \phi_- = \epsilon\frac{1}{2}i\phi_- \tag{113}$$

The tensors ψ are eigen vectors of the operator : $\left(\sum_{a=1}^3 r^a [\kappa_a]\right) \otimes (1_U)$ in the following meaning :

$\left(\sum_{a=1}^3 r^a [\kappa_a]\right) \otimes (1_U) (\psi) = \left(\sum_{a=1}^3 r^a [\kappa_a]\right) \otimes (1_U) (\phi_+ \otimes \sigma_+) + \left(\sum_{a=1}^3 r^a [\kappa_a]\right) \otimes (1_G) (\phi_- \otimes \sigma_-)$

$= \left(\sum_{a=1}^3 r^a [\kappa_a]\right) (\phi_+) \otimes (1_U) (\sigma_+) + \left(\sum_{a=1}^3 r^a [\kappa_a]\right) (\phi_-) \otimes (1_U) (\sigma_-)$

$= i\epsilon (\phi_+) \otimes (1_U) (\sigma_+) + i\epsilon (\phi_-) \otimes (1_U) (\sigma_-) = \frac{1}{2}i\epsilon\psi$

Under a change of gauge these states tensors transform as usual. If the change is of the kind : $(\rho \circ \Upsilon (\exp \tau \sum r_a^a \overrightarrow{\kappa}), 1_U)$ we get:

$\widehat{\psi} = \vartheta (\rho \circ \Upsilon (\exp \tau \sum r_a^a \overrightarrow{\kappa}), 1_U) \psi = \left(\exp \epsilon\tau i\frac{1}{2}\right) \psi$

6) With the γ previously defined one can compute the matrices $[\kappa_a]$. Let us fix the real scalars (r_1, r_2, r_3) and compute the eigen vectors of the matrix $[\kappa_S] = \sum_a r^a [\kappa_a]$

We get the following with $\sum_{j=1}^3 (r_j)^2 = 1$:

a) If $r_3 \neq 1$:

eigen value : $\epsilon\frac{1}{2}i : \phi_- = \begin{bmatrix} 0 \\ 0 \\ \epsilon(-r_1 + ir_2) \\ -1 + \epsilon r_3 \end{bmatrix} \in F^-, \phi_+ = \begin{bmatrix} \epsilon(-r_1 + ir_2) \\ -1 + \epsilon r_3 \\ 0 \\ 0 \end{bmatrix} \in F^+$,

b) If $r_3 = 1$:

eigen value : $\frac{1}{2}i : \phi_- = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \in F^-, \phi_+ = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \in F^+;$

eigen value : $-\frac{1}{2}i : \phi_- = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \in F^-, \phi_+ = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in F^+;$

Conversely for any vector of F such as : $\phi = \begin{bmatrix} ue^{i\alpha} \\ ve^{i\beta} \\ ue^{i\alpha} \\ ve^{i\beta} \end{bmatrix} \neq 0, u, v \in \mathbb{R} \Leftrightarrow$

$\phi_R = \phi_L$ one can find (r_1, r_2, r_3) such that ϕ is an eigen vector of $\sum_a r^a [\kappa_a]$:
$r_1 = -2\epsilon \frac{uv}{(u^2+v^2)} \cos(\nu)\,; r_2 = 2\epsilon \frac{uv}{(u^2+v^2)} \sin(\nu)\,; r_3 = -\frac{\epsilon\left(u^2-v^2\right)}{(u^2+v^2)}$

The set of such vectors is the set of invariant vectors under $-i\gamma_1\gamma_2\gamma_3 = \begin{bmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{bmatrix}$. It is a 2 dimensional subspace of F, but it is not invariant by the action of Spin(3,1). Indeed $-i\gamma_1\gamma_2\gamma_3 = -i\rho(\varepsilon_1\varepsilon_2\varepsilon_3)$ and there are elements of Spin(3,1) which do not commute with $\varepsilon_1\varepsilon_2\varepsilon_3$ as it is not too difficult to see in Cl(4,C). So if a state is symmetric for some frame, it is no longer symmetric for at least another frame : for these particles there are privileged frames, and so privileged directions in the universe.

The spatially symmetric states are : $\psi = \phi_+ \otimes \sigma_R + \phi_- \otimes \sigma_L$ in matrix form:

$[\psi] = \begin{bmatrix} \psi_R \\ \psi_L \end{bmatrix} = \begin{bmatrix} \phi\sigma_R \\ \phi\sigma_L \end{bmatrix}$ with $[\phi] = \begin{bmatrix} ue^{i\alpha} \\ ve^{i\beta} \end{bmatrix}$ 2x1 matrix, $[\sigma_R], [\sigma_L]$ 1xm matrix

7) It is easy to compute the moments with these states :

$[\phi]^* [\phi] = u^2 + v^2$

$[\phi]^* \sigma_1 [\phi] = 2uv\cos(\nu) = -\epsilon r_1 (u^2 + v^2)$

$[\phi]^* \sigma_2 [\phi] = -2uv\sin(\nu) = -\epsilon r_2 (u^2 + v^2)$

$[\phi]^* \sigma_3 [\phi] = u^2 - v^2 = -\epsilon r_3 (u^2 + v^2)$

$\sum_1^3 \left([\phi]^* \sigma_a [\phi]\right)^2 = (u^2 + v^2)^2$

All these quantities depend on 3 real scalars only. One cannot have $[\phi]^* \sigma_0 [\phi] = 0$ or r>0: $[\phi]^* \sigma_r [\phi] = 0$

With the formulas given previously the moments are the following :

a)

$a < 4 : P_a = -\epsilon r_a (u^2 + v^2) \operatorname{Im}\left([\sigma_L]^t \overline{[\sigma_R]}\right)$

$a > 3 : P_a = \epsilon r_{a-3} (u^2 + v^2) \operatorname{Re}\left([\sigma_L]^t \overline{[\sigma_R]}\right)$

b)

$J_0 = -\frac{1}{2}(u^2 + v^2)\left([\sigma_R]^t \overline{[\sigma_R]} + [\sigma_L]^t \overline{[\sigma_L]}\right)$;

$r > 0 : J_r = \frac{1}{2}(u^2 + v^2)\,\epsilon r_r \left([\sigma_R]^t \overline{[\sigma_R]} - [\sigma_L]^t \overline{[\sigma_L]}\right)$

$\sum_0^3 J_r^2 = \frac{1}{2}(u^2 + v^2)^2 \left(\left([\sigma_R]^t \overline{[\sigma_R]}\right)^2 + \left([\sigma_L]^t \overline{[\sigma_L]}\right)^2\right) > 0$

c) $\langle \psi, \psi \rangle = -2(u^2 + v^2) \operatorname{Im}\left([\sigma_L]^t \overline{[\sigma_R]}\right)$

d) $\rho^a = 2(u^2 + v^2) \operatorname{Re}\left([\sigma_L][\theta_a]^t [\sigma_R]^*\right)$

e)

$[\mu_A]_a^0 = -i(u^2 + v^2)\left([\sigma_R][\theta_a]^t [\sigma_R]^*\right) + \left([\sigma_L][\theta_a]^t [\sigma_L]^*\right)$

$[\mu_A]^j_a = i\epsilon r_j \left(u^2+v^2\right)\left([\sigma_R][\theta_a]^t[\sigma_R]^*\right) - \left([\sigma_L][\theta_a]^t[\sigma_L]^*\right)$

The kinematic moments depend on 3 real scalars (for ϕ) and 2 real $\left([\sigma_R][\sigma_R]^*, [\sigma_L][\sigma_L]^*\right)$ and 1 complex scalar $\left([\sigma_L][\sigma_R]^*\right)$ that is 7 degrees of freedom. The physical moments depend on 3 real scalars (for ϕ) and mx2 reals scalars $\left([\sigma_R][\theta_a]^t[\sigma_R]^*, \left([\sigma_L][\theta_a]^t[\sigma_L]^*\right)\right.$ imaginary) and m complex numbers $\left([\sigma_R][\theta_a]^t[\sigma_L]^*\right)$.

The moments are gauge and chart invariant, so their value is the same for all the observers.

19.3 Physical meaning

1) Let us pause to think about the physical meaning of these results. Starting from general assumptions, we have proven that the only spatial symmetry, as defined, are rotations around a space like vector r, and that the state tensor is then an eigen vector of an operator $\left(\sum_{a=1}^3 r^a[\kappa_a]\right)\otimes(1_U)$. But of course this outcome could be some mathematical artefact without physical meaning and only experiments could tell if such symmetries exist.

2) If particles have this kind of feature that means that for them some specific geometric directions are privileged, and we should expect that they behave accordingly. This is not an issue of quantization, but of the existence of something like an magnetic moment. And we know that it is the case for most of the elementary particles.

Indeed the equation 95 reads :

$N\varpi_4\left(a_D\rho^a\overrightarrow{V} - ia_I\overrightarrow{\mu}_a\right) = \nabla_e * \overline{\mathcal{F}}^a_A$

And for a symmetric state the "magnetic moment" is the 4-vector

$\overrightarrow{\mu} = \sum_r [\mu_A]^r_a \partial_r = \left(u^2+v^2\right)\sum_\alpha\left(O^\alpha_0\sigma^2 - \epsilon\left(r_1O^\alpha_1 + r_2O^\alpha_2 + r_3O^\alpha_3\right)\sigma^3\right)\partial_\alpha$

with :

$\sigma^1 = 2\,\mathrm{Im}\left([\sigma_R][\theta_a]^t[\sigma_L]^*\right)$

$i\sigma^2 = \left([\sigma_R][\theta_a]^t[\sigma_R]^*\right) + \left([\sigma_L][\theta_a]^t[\sigma_L]^*\right)$

$i\sigma^3 = \left([\sigma_R][\theta_a]^t[\sigma_R]^*\right) - \left([\sigma_L][\theta_a]^t[\sigma_L]^*\right)$

This equation is geometric : all the quantities are defined as vectors or tensors so, as written, it stands for any observer, it is fully gauge equivariant and covariant. The moments do not depend of the gauge, so the quantities r_1, r_2, r_3 are not the components of a vector expressed in a frame : they are invariant (it is the consequence of the invariance of the scalar product). The magnetic moment is defined as the *sum of vectors* :

$\overrightarrow{\mu} = (u^2+v^2)\sum_\alpha \left(\sigma^2\overrightarrow{\partial_0} - \epsilon\left(r_1\overrightarrow{\partial_1} + r_2\overrightarrow{\partial_2} + r_3\overrightarrow{\partial_3}\right)\sigma^3\right)$

If one changes the tetrad the vectors $\left(\overrightarrow{\partial}_i\right)$ change and $\overrightarrow{\mu}$ changes accordingly. This feature is common to all the moments that we have defined.

3) So it seems that we have a paradox, or an inconsistency : the vector $\overrightarrow{\mu}$ plays clearly some specific role, it is a well defined vector, but it changes with the observer.

Let us first clarify a point about the previous demonstration. We have characterized the set of observers O_S , but all the reasoning above has been done from a frame belonging to O_S , so if we could know the components of the vector r we do not know in which frame they are measured. Indeed our question was : "what are the symmetric states in a tetrad ?" and the answer is right, but it is true for any tetrad, and it is what we checked for the magnetic moment.

So from the above equation it is probable that any observer can find tetrads in which the states are symmetric, and check that the direction of the 4-vector $\overrightarrow{\mu}$ is specific, but these directions change with the observer. That the symmetry is geometrically defined does not entail that some particles, and over more their kinematic mode, have a symmetry of a geometric nature (which is anyway difficult to apprehend for pointlike objects).

4) Furthermore we do have a privileged orientation in our model : the velocity of the particles. As one can see in the equation above the net impact of the field depends of a sum of $\overrightarrow{V}$ and $\overrightarrow{\mu}_a$ and, even if the real and imaginary parts of the fields act differently, the net impact should depend of the relative orientations of the two vectors. The spin of a particle is measured with respect to its velocity. It would be of interest to investigate the equations listed here for particles which have moments of the symmetric mode.

20 PHYSICAL SYMMETRIES

One of the main features of particles is that their "physical characteristics", modelled in the space vector W, are constant. So we are lead to study the symmetries occuring in relation with the group U. They come in two flavours :

- families of particles sharing the same constant physical characteristics, and corresponding to subgroups of U : they define elementary particles

- specificities of the vacuum which force the states of the particles, as we see them, to belong to some representation of a subgroup of U. This is the symmetry breackdown mechanism

20.1 Families of particles

The simplest way to define families of particles is to proceed as above and look for symmetries related to the U group.

1) Two states ψ, ψ' are physically imperceptible if there are a vector subspace $W_S \subset W$, and a closed subgroup $U_S \subset U$ such that :

$\forall \psi \in F \otimes W_S, \forall s \in Spin(3,1), \forall u \in U_S, \exists \theta \in \mathbb{R} : \vartheta (s,u) \psi = e^{i\theta} \vartheta (s,1) \psi$

θ can depend on s, u and ψ.

2) Let be the 4 vectors σ_i of W such that : $\sigma_i = \sum_{j=1}^m \psi^{ij} f_j$ so $\psi = \sum_{i=1}^4 e_i \otimes \sigma_i$

So if $\psi \in F \otimes W_S : \forall s \in Spin(3,1), \forall u \in U_S$:

$\sum_{i=1}^4 \vartheta (s,u) (e_i \otimes \sigma_i) = \sum_{i=1}^4 e^{i\theta(u)} e_i \otimes \sigma_i = \sum_{i=1}^4 (\rho \circ \Upsilon (s) e_i) \otimes (\chi (u) \sigma_i)$

Let s=1 : $\sum_{i=1}^4 \vartheta (1,u) (e_i \otimes \sigma_i) = \sum_{i=1}^4 e_i \otimes (\chi (u) \sigma_i) = \sum_{i=1}^4 e_i \otimes e^{i\theta(u)} \sigma_i$

$\Rightarrow \forall \psi \in F \otimes W_S, \forall u \in U_S : \chi (u) \sigma_i = e^{i\theta(u)} \sigma_i$

So for each i $(\sigma_i, \chi|_{U_S})$ is a linear representation of U_S , of complex dimension 1. $(\mathbb{C}, \chi' (1))$ is a linear 1-dimensional representation of the Lie algebra of U_S. The only compact groups with non trivial 1 dimensional representations are abelian, so U_S is an abelian subgroup of U.

3) Any compact Lie group has abelian subgroups, which are tori. The maximal tori are p dimensional with p=rank of U, all conjugated to each others. One can always choose a basis in $T_1 U$ such that the first p vectors $\overrightarrow{\theta_j}$ belong to a maximal torus of the algebra. They are orthonormal in the hermitian, Ad invariant, scalar product on $T_1^c U$.

The irreducible representations $(W_\lambda, \chi_\lambda)$ of the compact group U are indexed on the highest weight λ, The corresponding vector space W_λ contains a vector σ_λ which is an eigen-vector of $\chi_\lambda (u)$ for any element of a maximal torus U_S:

$\forall u \in U_S : \chi_\lambda (u) \sigma_\lambda = e^{i\lambda(u)} \sigma_\lambda, \lambda (u) \in \mathbb{R}$

The other vectors of W_λ are generated as successive applications of some elements of G. So these vectors σ_λ characterize distinct families of particles.

Any representation of U is the sum of irreducible representations, so W has a collection of such vectors u_λ.

4) For a particle belonging to a family one should expect that its tensor state ψ is such that its physical components σ_λ stays within W_λ. But one cannot exclude more complicated states, involving more than one family of particles.

So the whole story is about the definition of irreducible representations of compact groups, and finding, through experiments, the representations which are found in the real world. This is at the foundation of gauge theories of particles physics.

21 SYMMETRY BREAKDOWN

21.0.1 Principle

1) The invariance of the lagrangian implies that the potentials $\grave{A}$ must factorize through the covariant derivative, or the curvature form for their derivatives. In quantum theory of fields a particle - a boson - must be associated to each field and therefore this boson must be massless, contradictory to the experiments. Symmetry breakdown is first a mechanism to turn over this issue, so far critical to the consistency of the standard model of particles.

2) The "Higgs mechanism" can be summarized as follow (Bednyakov [2]). The Euler Lagrange equations give only necessary conditions, and it could happen that the solutions are not unique. This can be a mathematical artefact but, in many physical situations, a system can possibly follow several paths, and the actual choice depends on the initial conditions or an outside action. In these cases it is logical to reparametrize the model, generally by introducing discrepancy variables. In the Higgs mechanism it is the fundamental state, corresponding to the vacuum, which is assumed to offer several paths (the vacuum state is degenerate). The most common explanation to this phenomenon is cosmological : the specificities of the present vacuum would come from the initial conditions of the "big bang". Some kinds of particles would have been privileged, and the situation hereafter would have

been frozen, as if a phase transition had occured. Thus to account for this specific initial values conditions, one proceeds to a change of variables, evidencing the discrepancy with the actual vacuum. The new variables take the form of fields (Goldstone bosons and Higgs field) which interact with the existing matter and fields and give a mass to some bosons. This correction to the gauge model is phenomenological : we see one privileged solution among others, and the basic theory cannot forecast which one the system takes, so additional variables are needed, that only experiments can fixed.

3) In particle physics all this happens in the framework of quantum theory of fields, but the breakdown of symmetry is actually a fairly common phenomenon, met in classical situations, such as ferromagnetism or phase transition, and therefore it is in the scope of classical field theory. In the general picture used here:

a) H is a subgroup of the group U of "internal symmetries" (the kinematic part is not involved here). A member of U can be written as $u = xh$ with $x \in X = U/H$. The quotient space X acts as an intermediary level in the gauge group and there is a fiber bundle X_M over M modelled on U/H.

b) The system is still modelled as previously but there is some "fundamental state" of the universe, pre-existing to any system, and characterized by a section $\varkappa(m)$ of the fiber bundle X_M, similar to the "Higgs field". This field interacts with the force fields (other than gravitation) and therefore constrains their value and conversely this interaction fixes $\varkappa$. A transition phase has occurred.

c) Therefore the gauge group is reduced in that the only visible gauge transformations are those in the equivalence class of $\varkappa$, that is of the form $u = xh$, with H as apparent gauge group : the action of U is "hidden" by the Higgs field. The states ψ are still described in a fiber bundle associated to U, and the principle of least action still stands, but one has to account for the pre-existing Higgs field $\varkappa$

So, this is not simple...I found it better to proceed step by step, and use long but basic mathematical developments than to race through highly specialized short-cuts. We have to address successively the additional mathematical structures for U, the fiber bundle U_M, the connection $\mathbf{A}$, and the fiber bundle of fields.

21.0.2 The fiber bundle U

The basic rule is that any element in U can be written as : $u = \Lambda(x,h)$ where $x \in U/H, h \in H$

1) H is assumed to be a closed non discret subgroup of U, it is therefore a compact Lie group. The quotient space X=U/H (called homogeneous space) is defined by the equivalence relation :

$x \sim y \in U \Leftrightarrow \exists h \in H : x = yh \Leftrightarrow y^{-1}x \in H$

2) Under this assumption U is a *principal fiber bundle over U/H of group H* (Kolar [ ] 10.5) which implies :

a) U/H is a smooth metrisable manifold

b) There is a projection $\pi_H : U \to U/H$ such that $\forall s \sim s' \in U : \forall u \in U : \pi_H(us) = \pi_H(us')$ and therefore :

$h \in H : \pi_H(h) = \pi_H(1_U) = \pi_H(1_H)$

π_H is onto : $\forall \xi_x \in T_xU/H, \exists \overrightarrow{\theta} \in T_1U : \pi'_H(x)\overrightarrow{\theta} = \xi_x$

c) There is an open cover (U_i) of U and trivializations :

$\Lambda_i : \pi_H^{-1}(U_i) \times H \to U :: \Lambda_i(x,h) \in U$ with the usual *right* action of H on U :$\Lambda_i(x,h) \times h' = \Lambda_i(x,hh')$

The trivializations are defined by the values $\Lambda_i(x,1)$

d) The fundamental vectors $\Lambda'_h(x,h)R'_h(1)\xi^h, \xi^h \in T_1H$ are the generators of the vertical space isomorphic to T_1H and:

$\pi'_H(\Lambda_i(x,h))\Lambda'_{ih}(x,h)R'_h(1)\xi^h = 0$

$\pi_H(\Lambda_i(x,h)) = x \Rightarrow \pi'_H(\Lambda_i(x,h))\Lambda'_{ix}(x,h)\xi^x = \xi^x$

e) There is a *left* action of U on U/H which is denoted $\lambda(u,x)$:

$\pi_H(s) = x : \lambda(u,x) = \pi_H(us) \Leftrightarrow \lambda(u,\pi_H(s)) = \pi_H(us)$

2) In the standard model U is the direct product of the compact groups $U = SU(3) \times SU(2) \times U(1)$ and H is the projection of U on SU(2)xU(1) or U(1). Without being too specific but with the purpose to be simple we will assume the following:

a) $T_1U = h_0 \oplus l_0$ with h_0 a r dimensional sub-algebra and l_0 a m-r vector subspace. The basis of T_1U is comprised of r vectors $\overrightarrow{\theta}_a \in h_0$ (a=1..r) and m-r vectors $\overrightarrow{\theta}_a \in l_0$ (a=r+1...m)

b) h_0 is the Lie algebra of the compact subgroup H, which is generated by exp $\left(\sum_{a=1}^{r} r^a \overrightarrow{\theta}_a\right)$

c) The map : $H \times l_0 \to U :: u = \exp l \times h$ is a diffeomorphism. One can identify x with $\Lambda(x,1)$,X=U/H with a subset of U : $X = U/H = \{\exp l, l \in l_0\}$ and $\Lambda(x,h) = \Lambda(\exp l, h) = (\exp l)\, h$. The fiber bundle $U(X, H, \pi_H)$ is trivial.

d) The bracket on T_1G is such as : $[h_0, h_0] \subset h_0, [l_0, l_0] \subset h_0, [h_0, l_0] \subset l_0$

e) There is a bilinear symmetric scalar form on T_1U, invariant by the adjoint operator, for which the subspaces h_0, l_0 are oorthogonal, and positive definite on T_1H.

These conditions are met if U and H are linked in a Cartan decomposition (Knapp [ ] 6.31). All semi-simple Lie groups have such decompositions. The conditions d) and e) will not be used in the following but are part of the definition of a Cartan decomposition.

Remark : property c is usually written as: $u = h \exp l$. Both formulations are equivalent.

Proof. Indeed : ■

$Ad_h(h_0) \subset h_0$ because h_0 is the Lie algebra of H, h_0 is Ad_h invariant, so is its orthogonal complement l_0.

Thus: $\forall h \in H, l \in l_0, \exists l' \in l_0 : Ad_h l = l' \Rightarrow \exp Ad_h l = h(\exp l)\, h^{-1} = \exp l' \Rightarrow \exp(-r) \exp l' \exp r = \exp l$

$u = \exp r \exp l = \exp r \exp(-r) \exp l' \exp r = \exp l' \exp r$■

21.0.3 Principal fiber bundles on M

The splitting U/H, H is prolonged in the principal fiber bundles on M. The principal fiber bundle $U_M \to M$ split in : $X_M \to M$ corresponding to U/H and $\widetilde{U}_M \to X_M$ corresponding to H. The splitting is attributable to the Higgs field, materialized by a section $\varkappa$ on X_M.

1) We still have the same principal fiber bundle U_M base M, group U, with the projection $\pi : U_M \to M$, and the trivializations on an open cover: $\varphi_{Ui} : \pi^{-1}(U_{Mi}) \times U \to U_M :: p = \varphi_{Ui}(m,u)$ and we denote $\widehat{p_i} = \varphi_{Ui}(m,1)$,so $p = u_i \widehat{p}_i = \widehat{p}_i u_i$

As a manifold U_M has charts deduced from $\varphi_{Ui}(m, \Lambda(x,h)) = p$ and one can construct additional structures.

2) The *associated fiber bundle* $X_M = U_M \times_U X$ with typical fiber X=U/H associated with U_M through the U action :

$(U_M \times X) \times G \to (U_M \times X) :: (p,x) \times u \to (pu^{-1}, \lambda(u,x))$

X_M is a U-fiber bundle (but not a principal fiber bundle), with base M, trivializations:

$\pi^{-1}(U_{Mi}) \times U/H \to X_M : \varphi_{Xi}(m,x) = \varphi_{Ui}(m, \Lambda(x,1))$

and U left action : $u \times \varphi_{Xi}(m,x) = \varphi_{Xi}(m, \lambda(u,x))$

With a Cartan decomposition X_M can be seen as a sub-bundle, embedded in $U_M : \varphi_{Ui}(m, \exp l)$

A section on X_M is a map : $\chi(m) = \varphi_{Xi}(m, \varkappa(m)) = \varphi_{Ui}(m, \Lambda(x(m), 1))$ .The Higgs fields are such sections :they fix the state of the vacuum, which is characterized by an equivalence class of U/H. With the Cartan decomposition : $\chi(m) = \varphi_{Ui}(m, \exp l(m))$ where $l : M \to l_0$

3) U_M is endowed with the *principal fiber bundle* structure $\widetilde{U}_M$ *with base* X_M *, group* H and :

projection : $\widetilde{\pi}(\varphi_{iU}(m,u)) = \varphi_{iX}(m, \pi_H(u))$

open cover : $X_{Mi} = U_{Mi} \cap X_M$

trivializations : $\widetilde{\varphi}_{Ui} : X_{Mi} \times H \to U_M :: \widetilde{\varphi}_{Ui}(q,h) = qh$

$\Leftrightarrow \widetilde{\varphi}_{Ui}(\varphi_{Ui}(m, \Lambda(x,1)), h) = \varphi_{Ui}(m, \Lambda(x,h))$

H action : $h \times p = h \times \widetilde{\varphi}_{Ui}(q,h') = \varphi_{Ui}(\pi(p), \Lambda(x, hh'))$

A section on $\widetilde{U}_M$ is : $s = \widetilde{\varphi}_U(q, h_s(q))$

One has a 2-levels composite fiber bundle : $U_M \equiv \widetilde{U}_M \xrightarrow{H} X_M \xrightarrow{U} M$

4) The composition of a section $\varkappa$ on X_M and a section s on $\widetilde{U}_M$ is a section S on U_M. Conversely a section s on $\widetilde{U}_M$ has for image $s(X_M)$ a sub-manifold embedded in U_M. Any section S in U_M is the composite of $\varkappa(m) = \widetilde{\pi}(S(m))$ on X_M and a section s on $\widetilde{U}_M : M \xrightarrow{\varkappa} X_M \xrightarrow{s} U_M : S(m) = s \circ \varkappa(m)$. Remember that the physical characteristics σ are sections of U_M. They are now defined in two steps : the first with $\varkappa$, the second with H.

A local gauge transformation is given either by a section on U_M or by the composite:

$S(m) = s \circ \varkappa(m) = \widetilde{\varphi}_U(\varkappa(m), h_s(m)) ; \widetilde{\pi}(S(m)) = \varkappa(m)$

With Cartan decomposition : $S(m) = \varphi_U(m, \exp l(m) \exp r(m))$

5) There is a bijection between the *principal bundle structures* H_M *with base* M *and group* H on one hand, and the global sections $\varkappa(m)$ on X_M (Kolār [14] 10.13) : $\varkappa = \varphi_X(m, x(m))$. The trivialization is : $\varphi_H(m,h) = \widetilde{\varphi}_U(\varkappa(m), h) = \varphi_U(m, \Lambda(x(m), h))$. With Cartan decomposition : $\varkappa = \varphi_G(m, \exp l(m)) \to \varphi_H(m,h) = \varphi_U(m, \exp l(m) \times h)$.

Such H_M fiber bundles are not necessarily isomorphic. So its definition requires both U_M and χ.

21.0.4 The induced connections

A connection is a projection from the tangent space on the vertical space. The tangent space of U_M splits and a connection **A** induces a connection on X_M, but it induces a connection on H_M iff $\nabla\varkappa = 0$. Let us first define the tangent spaces.

1)

a) The tangent space of $U \to U/H$ can be defined through the trivialization :

$(\xi^x, \xi^h) \in T_xX \times T_hH \to \overrightarrow{\xi}^u = \Lambda'(x,h)(\xi^x, \xi^h) \in T_uU,$

$\overrightarrow{\xi}^u = \Lambda'_x(x,h)\xi^x + \Lambda'_h(x,h)\xi^h = (R'_hx)\xi^x + (L'_xh)\xi^h \Leftrightarrow \Lambda'_h(x,h) = L'_xh; \Lambda'_x(x,h) = R'_hx$

X=U/H can be identified with a subset of U, so the tangent space T_uU splits :

$T_uU = \{L'_uT_1U\} = \{R'_uT_1U\}$ Notice that the map works on right and left, we will need both

$\overrightarrow{\theta}^h = \sum_{a=1}^r l^a\widehat{\theta}_a \in h_0 \to \xi^h = (L'_h1)\overrightarrow{\theta}^h,$

$\overrightarrow{\theta}^l = \sum_{a=r+1}^m l^a\widehat{\theta}_a \in l_0 \to \xi^x = (R'_x1)\overrightarrow{\theta}^l,$

$\overrightarrow{\xi}^u = (R'_hx)(R'_x1)\overrightarrow{\theta}^l + (L'_xh)(L'_h1)\overrightarrow{\theta}^h = R'_u1\overrightarrow{\theta}^l + L'_u1\overrightarrow{\theta}^h$

$\overrightarrow{\xi}^u = L'_u1\overrightarrow{\theta}' = R'_u1\overrightarrow{\theta}" \Rightarrow \overrightarrow{\theta}" = R'_{u^{-1}}(u)L'_u1\overrightarrow{\theta}' = Ad_u\overrightarrow{\theta}'$

The vectors $(L'_u1)\overrightarrow{\theta}^h, \overrightarrow{\theta}^h \in l_0$ are the generators of the vertical space :

$\pi'_H(u)(L'_u1)\overrightarrow{\theta}^h = 0$

$\pi'_H(u)\overrightarrow{\xi}^u = R'_u1\overrightarrow{\theta}^l$

and we have $\lambda(u,x) = \pi_H(ux) \Rightarrow$

$\lambda'_u(u,x)\xi^u + \lambda'_x(u,x)\xi^x = \pi'_H(ux)(R'_xu\xi^u + L'_ux\xi^x)$

$\lambda'_u(u,x) = \pi'_H(ux)R'_xu;$

$\lambda'_x(u,x) = \pi'_H(ux)L'_ux$

b) The tangent space T_pU_M can be defined by :

$(\xi^m, \xi^u) \in T_mM \times T_uU \to \xi_p = \varphi'_{iUm}(m,u)\xi^m + \varphi'_{iUu}(m,u)\xi^u,$

The vertical space splits : $\varphi'_{iUu}(m,u)\xi^u = \varphi'_{iUu}(m,u) R'_u 1 \overrightarrow{\theta}^l + \varphi'_{iUu}(m,u) L'_u 1 \overrightarrow{\theta}^h$
and we have the basis :

a=1,..r : $\delta_a(p) = \varphi'_{iUu}(m,u) L'_u 1 \overrightarrow{\theta}_a$

a=r+1,...m: $\delta_a(p) = \varphi'_{iUu}(m,u) R'_u 1 \overrightarrow{\theta}_a$

c) The tangent space $T_q X_M$ can be defined by :

$q = \varphi_{iX}(m,x) = \varphi_{iU}(m,\Lambda(x,1))$

$\xi_q = \varphi'_{iX}(m,x)(\xi^m,\xi^x) = \varphi'_{iUm}(m,\Lambda(x,1))\xi^m + \varphi'_{iUu}(m,u) R'_x 1 \overrightarrow{\theta}^l$

the vertical space isomorphic to $T_x X$ is generated by : $\varphi'_{iUu}(m,u) R'_x 1 \overrightarrow{\theta}^l$

d) The tangent space $T_q \widetilde{U}_M$ can be defined by :

$q = \widetilde{\varphi}_{iU}(\varphi_{iU}(m,\Lambda(x,1)),h) = \varphi_{iU}(m,\Lambda(x,h))$

$\left(\xi^m,\xi^x,\xi^h\right) \in T_m M \times T_x X \times T_h H$

$\rightarrow \xi_q = \varphi'_{iUm}(m,\Lambda(x,1))\xi^m + \varphi'_{iUu}(m,\Lambda(x,1))\left(\Lambda'_x(x,h)\xi^x + \Lambda'_h(x,h)\xi^h\right)$

$= \varphi'_{Um}(m,\Lambda(x,1))\xi^m + \varphi'_{Uu}(m,\Lambda(x,1)) R'_h x \xi^x + \varphi'_{Uu}(m,\Lambda(x,1)) L'_x h \xi^h$

$= \varphi'_{Um}(m,\Lambda(x,1))\xi^m + \varphi'_{Uu}(m,\Lambda(x,1)) R'_{xh} 1 \overrightarrow{\theta}^l + \varphi'_{Uu}(m,\Lambda(x,1)) L'_{xh} 1 \overrightarrow{\theta}^h$

So it can be identified with $T_{\Lambda(x,h)} U_M$ but the vertical space is generated here by $\varphi'_{Uu}(m,\Lambda(x,1)) L'_{xh} 1 \overrightarrow{\theta}^h$.

e) The tangent space $T_q H_M$ with the section $\chi \in \Lambda_0 X_v$ can be defined by :

$q = \varphi_H(m,h) = \varphi_{Ui}(m,\Lambda(x(m),h))$

$\left(\xi^m,\xi^h\right) \in T_m M \times T_h H \rightarrow \xi_q = \varphi'_{Hm}(m,h)\xi^m + \varphi'_{Hh}(m,h)\xi^h$

$\xi_q = \varphi'_{Um}(m,\Lambda(\varkappa(m),h))\xi^m$

$+\varphi'_{Uu}(m,\Lambda(\varkappa(m),h))\left(\Lambda'_x(\varkappa(m),h)\varkappa'(m)\xi^m + \Lambda'_h(\varkappa(m),h)\xi^h\right)$

$= \left(\varphi'_{Um}(m,\Lambda(\varkappa(m),h)) + \varphi'_{Uu}(m,\Lambda(\varkappa(m),h))(R'_h\varkappa)\varkappa'(m)\right)\xi^m$

$+\varphi'_{Uu}(m,\Lambda(\varkappa(m),h))\left(L'_{\varkappa(m)}h\right)\xi^h$

$\varphi'_{Hh}(m,h)\xi^h = \varphi'_{Uu}(m,\Lambda(\varkappa(m),h))\left(L'_{\varkappa(m)}h\right)\xi^h$

The vertical space is generated by:

$\varphi'_{Hh}(m,h)\xi^h = \varphi'_{Uu}(m,\Lambda(\varkappa(m),h))\left(L'_{\varkappa(m)h}1\right)\overrightarrow{\theta}^h$

2)

a) We assume as above that there is a principal connection $\mathbf{A}$ on U_M with one-form $\widehat{\grave{A}}(p) = \sum_{a\alpha} \widehat{\grave{A}}_\alpha^a (p)\, dx^\alpha \otimes \overrightarrow{\theta}_a$ and potential $\grave{A}(m) = \widehat{\grave{A}}(\varphi_U(m, 1_U))$.

$\widehat{\grave{A}}(\varphi_G(m,u)) = Ad_{u^{-1}} \grave{A}(m)$

$\varphi_U^* \mathbf{A}(p)(\xi^m, \xi^u) = \varphi'_{Uu}(m,u)\left(\xi^u + R'_u(1)\grave{A}(m)\xi^m\right)$

It splits along the two subspaces h_0, l_0:

$\xi^u = R'_u 1 \overrightarrow{\theta}^l + L'_u 1 \overrightarrow{\theta}^h$

$\grave{A}(m)$

$= \sum_\alpha \sum_{a=1}^r \grave{A}_\alpha^a (m)\, dx^\alpha \otimes \overrightarrow{\theta}_a + \sum_\alpha \sum_{a=r+1}^m \grave{A}_\alpha^a (m)\, dx^\alpha \otimes \overrightarrow{\theta}_a$

$= \grave{A}_h + \grave{A}_l$

$\varphi_U^* \mathbf{A}(p)\left(\xi^m, (R'_u 1)\overrightarrow{\theta}^l + (L'_u 1)\overrightarrow{\theta}^h\right)$

$= \varphi'_{Uu}(m,u)\left((R'_u 1)\left(\overrightarrow{\theta}^l + \grave{A}_l \xi^m\right) + (L'_u 1)\left(\overrightarrow{\theta}^h + \grave{A}_h \xi^m\right)\right)$

b) $\mathbf{A}$ induces the linear (not equivariant) connection $\mathbf{\Gamma}$ on the associated fiber bundle X_M :

$\varphi_X^* \mathbf{\Gamma}(q)(\xi^m, \xi^x) = \varphi'_{Uu}(m, \Lambda(x,1))\left(\xi^x + \lambda'_u(1,x)\grave{A}(m)\xi^m\right)$

With $\lambda'_u(1,x) = \pi'_H(x) R'_x 1 \in L(T_1 U; T_x X)$

$\lambda'_u(1,x)\grave{A}(m)\xi^m$

$= \pi'_H(x)(R'_x 1)\left(\sum_\alpha \sum_{a=1}^m \grave{A}_\alpha^a \xi^{m,\alpha}\right)$

$= (R'_x 1)\left(\sum_\alpha \sum_{a=r+1}^m \grave{A}_\alpha^a \xi^{m,\alpha}\right)$

that is : $\Gamma(q) = (R'_x 1)\sum_{a=r+1}^m \grave{A}_\alpha^a (m)\, \overrightarrow{\theta}_a \otimes dx^\alpha$

The covariant derivative of a section $\varkappa = \varphi_X(m, x(m))$ is :

$\nabla^X \varkappa = \varkappa^* \mathbf{\Gamma}(q) = \varphi'_{Uu}(m, \Lambda(x,1))\left(\frac{dx}{dm} + \lambda'_u(1, x(m))\grave{A}(m)\right)$

c) A connection Γ_H on H_M is a projection on the vertical space isomorphic to $T_h H$:

$q = \varphi_H(m,h) = \varphi_U(m, \Lambda(x(m), h))$

$\varphi_H^* \mathbf{\Gamma}_H(q)\left(\xi^m, \xi^h\right) = \varphi'_{Hh}(m,h)\left(\xi^h + \Gamma_H(q)\xi^m\right), \Gamma_H(q) \in \Lambda(T_m M; T_h H)$

It is principal iff : $\Gamma_H(\varphi_H(m,h)) = Ad_{h^{-1}}\Gamma_H(\varphi_H(m,1)) = Ad_{h^{-1}}\Gamma_H(\varphi_U(m, \varkappa(m)))$

that is $\mathbf{\Gamma}_H(q)\xi_q = \varphi'_{Hh}(m,h)\left(\xi^h + (R'_h 1)\Gamma_H(\chi)\xi^m\right)$

A connection $\mathbf{A}$ induces a principal connection on H_M iff $\nabla^X \varkappa = 0$

Proof. : ∎

$\varkappa(m) = \Lambda(\varkappa(m), 1), u = \Lambda(\varkappa(m), h), p = \varphi_{Ui}(m,u)$

$\xi_q = \varphi'_{Hm}(m,h)\xi^m + \varphi'_{Hh}(m,h)\xi^h$

$= \left(\varphi'_{Um}(m,u) + \varphi'_{Uu}(m,u)(R'_h\varkappa)\varkappa'(m)\right)\xi^m + \varphi'_{Uu}(m,u)\left(L'_{\varkappa(m)}h\right)\xi^h$
$\xi_q \in T_pU_M$ so one can compute :
$\varphi^*_U\mathbf{A}(p)\left(\xi^m,(R'_h\varkappa)\varkappa'\xi^m + (L'_\varkappa h)\xi^h\right)$
$= \varphi'_{Uu}(m,u)\left((R'_h\varkappa)\varkappa'\xi^m + (L'_\varkappa h)\xi^h + (R'_u1)\grave{A}(m)\xi^m\right)$
$(R'_u1)\grave{A}(m)\xi^m = (R'_h\varkappa)(R'_\varkappa 1)\grave{A}(m)\xi^m$
$(R'_\varkappa 1)\grave{A}(m)\xi^m \in T_\varkappa U$
$\Rightarrow \exists \overrightarrow{\theta}^h, \overrightarrow{\theta}^l : (R'_\varkappa 1)\grave{A}(m)\xi^m = R'_\varkappa 1\overrightarrow{\theta}^l + L'_\varkappa 1\overrightarrow{\theta}^h$
$\pi'_H(\varkappa)(R'_\varkappa 1)\grave{A}(m)\xi^m = \pi'_H(\varkappa)R'_\varkappa 1\overrightarrow{\theta}^l + \pi'_H(\varkappa)L'_\varkappa 1\overrightarrow{\theta}^h = R'_\varkappa 1\overrightarrow{\theta}^l$
but $\lambda'_u(1,\varkappa) = \pi'_H(\varkappa)R'_x1$ so one can write:
$R'_\varkappa 1\overrightarrow{\theta}^l = \lambda'_g(1,\varkappa)\grave{A}(m)\xi^m \Rightarrow L'_\varkappa 1\overrightarrow{\theta}^h = (R'_\varkappa 1)\grave{A}(m)\xi^m - \lambda'_g(1,\varkappa)\grave{A}(m)\xi^m$
$\overrightarrow{\theta}^h = \left(L'_{\varkappa^{-1}}\varkappa\right)\left((R'_\varkappa 1) - \lambda'_g(1,\varkappa)\right)\grave{A}(m)\xi^m$
$\overrightarrow{\theta}^h$ is a linear function of ξ^m, which does not depend on h (but on $\varkappa$).
Let us define :
$\Gamma_H(\varphi_H(m,1)) : T_mM \to T_1H :: \Gamma_H(\chi(m)) = \left(L'_{\varkappa^{-1}}\varkappa\right)\left((R'_\varkappa 1) - \lambda'_g(1,\varkappa)\right)\grave{A}(m)$
$(R'_u1)\grave{A}(m)\xi^m = (R'_h\varkappa)(R'_\varkappa 1)\grave{A}(m)\xi^m = (R'_h\varkappa)\left(\lambda'_u(1,\varkappa)\grave{A}(m) + (L'_\varkappa 1)\Gamma_H(\chi)\right)\xi^m$
and going back to :
$\varphi^*_U\mathbf{A}(p)\left(\xi^m,(R'_h\varkappa)\varkappa'\xi^m + (L'_\varkappa h)\xi^h\right)$
$= \varphi'_{Uu}(m,u)\left((R'_h\varkappa)\varkappa'\xi^m + (L'_\varkappa h)\xi^h + (R'_h\varkappa)\left(\lambda'_u(1,\varkappa)\grave{A}(m) + (L'_\varkappa 1)\Gamma_H(\chi)\right)\xi^m\right)$
$= \varphi'_{Uu}(m,u)\left((R'_h\varkappa)\left(\varkappa' + \lambda'_u(1,\varkappa)\grave{A}(m)\right)\xi^m\right)$
$+\varphi'_{Uu}(m,u)\left((L'_\varkappa h)\xi^h + (R'_h\varkappa)(L'_\varkappa 1)\Gamma_H(\chi)\xi^m\right)$
$\varphi'_{Uu}(m,\varkappa)\left(\varkappa' + \lambda'_u(1,x(m))\grave{A}(m)\right)\xi^m = \left(\nabla^X\varkappa\right)\xi^m$
$\left((L'_\varkappa h)\xi^h + (R'_h\varkappa)(L'_\varkappa 1)\Gamma_H(\chi)\xi^m\right) = (L'_\varkappa h)\left(\xi^h + R'_h(1)\Gamma_H(\chi)\xi^m\right)$
with : $R'_h(\varkappa) = R'_{\varkappa h}(1)R'_{\varkappa^{-1}}(\varkappa) = L'_{\varkappa h}(1)Ad_{(\varkappa h)^{-1}}R'_{\varkappa^{-1}}(\varkappa)$
$= L'_g(1)Ad_{h^{-1}}Ad_{\varkappa^{-1}}R'_{\varkappa^{-1}}(\varkappa) = L'_u(1)L'_{h^{-1}}(h)R'_h(1)L'_{\varkappa^{-1}}(\varkappa)R'_\varkappa(1)R'_{\varkappa^{-1}}(\varkappa)$
$= L'_\varkappa(h)R'_h(1)L'_{\varkappa^{-1}}(\varkappa)$
$\varphi'_{Uu}(m,u)(L'_\varkappa h)\left(\xi^h + R'_h(1)\Gamma_H(\chi)\xi^m\right) = \varphi'_{Hh}(m,h)\left(\xi^h + R'_h(1)\Gamma_H(\chi)\xi^m\right)$
$\varphi^*_U\mathbf{A}(p)\left(\xi^m,(R'_h\varkappa)\varkappa'\xi^m + (L'_\varkappa h)\xi^h\right)$
$= \left(\nabla^X\varkappa\right)\xi^m + \varphi'_{Hh}(m,h)\left(\xi^h + R'_h(1)\Gamma_H(\chi)\xi^m\right)$
If $\left(\nabla^X\varkappa\right) = 0$ then we have the principal connection
$\varphi'_{Hh}(m,h)\left(\xi^h + R'_h(1)\Gamma_H(\chi)\xi^m\right)$
This condition is also necessary (Giachetta [5] 5.10.5).■
The potential of $\mathbf{\Gamma}_H$ is :
$\Gamma_H(\chi(m)) = \left(L'_{\varkappa^{-1}}\varkappa\right)\left((R'_\varkappa 1) - \lambda'_u(1,\varkappa)\right)\grave{A}(m)$

$= Ad_{\varkappa^{-1}}\grave{A}(m) - \left(L'_{\varkappa^{-1}}\varkappa\right) R'_x 1 \left(\sum_\alpha \sum_{a=r+1}^m \grave{A}^a_\alpha \overrightarrow{\theta}_a \otimes dx^\alpha\right)$

$\Gamma_H(\chi(m)) = Ad_{\varkappa^{-1}}\left(\sum_\alpha \sum_{a=1}^r \grave{A}^a_\alpha \overrightarrow{\theta}_a \otimes dx^\alpha\right)$

3) The condition $\nabla^X \varkappa = 0$ reads :

$\frac{dx}{dm} + \lambda'_u(1, x(m))\grave{A}(m) = 0$

$\Rightarrow \frac{dx}{dm} = -\left(R'_{x(m)}1\right)\left(\sum_\alpha \sum_{a=r+1}^m \grave{A}^a_\alpha \overrightarrow{\theta}_a \otimes dx^\alpha\right)$

With $\varkappa(m) = \exp l(m)$ the first step to solve the problem is to find a map $l : M \to l_0$ such that :

$\left(\frac{d}{dl}\exp l\right)\frac{dl}{dm} = -R'_{x(m)}1\left(\sum_\alpha \sum_{a=r+1}^m \grave{A}^a_\alpha \overrightarrow{\theta}_a \otimes dx^\alpha\right)$

Using the derivative of exp (Duistermaat [4] 1.5):

$\left(\frac{d}{dl}\exp l\right)\frac{dl}{dm} = \left(R'_{x(m)}1\right)\left(\int_0^1 e^{\tau adl(m)} d\zeta\right)\frac{dl}{dm}$

$= -R'_{x(m)}1\left(\sum_\alpha \sum_{a=r+1}^m \grave{A}^a_\alpha \overrightarrow{\theta}_a \otimes dx^\alpha\right)$

So the map l(m) is solution of the equation :

$$\left(\int_0^1 e^{\tau adl(m)} d\tau\right)\frac{dl}{dm} = -\sum_\alpha \sum_{a=r+1}^m \grave{A}^a_\alpha \overrightarrow{\theta}_a \otimes dx^\alpha \qquad (114)$$

The map $\overrightarrow{\theta} \to \int_0^1 e^{\tau ad\overrightarrow{\theta}} d\tau$ is inversible if ad($\overrightarrow{\theta}$) is inversible. It is inversible and analytic in a neighbourood of 0 and its inverse is :

$\sum_{k=0}^\infty \frac{B_k}{k!}\left(ad\overrightarrow{\theta}\right)^k$ where B_k are the Bernouilli numbers.

$\int_0^1 e^{\tau adl} d\tau = \sum_{k=0}^\infty \int_0^1 \frac{1}{k!}\tau^k (adl(m))^k d\tau = \sum_{k=0}^\infty \left(\int_0^1 \tau^k d\tau\right)\frac{1}{k!}(adl(m))^k = \sum_{k=1}^\infty \frac{1}{k!}(adl(m))^{k-1}$

$\left(\int_0^1 e^{\tau adl(m)} d\tau\right)\frac{dl}{dm} = \sum_{k=0}^\infty \frac{1}{k!}\frac{1}{k}\frac{d}{dm}(adl(m))^k = \frac{d}{dm}\left(\sum_{k=1}^\infty \frac{1}{k!}\frac{1}{k}(adl(m))^k\right)$

Thus : $\frac{d}{d\xi^\alpha}{}_\alpha\left(\sum_{k=1}^\infty \frac{1}{k!}\frac{1}{k}(adl(m))^k\right) = -\sum_{a=r+1}^m \grave{A}^a_\alpha \overrightarrow{\theta}_a$

21.0.5 The Higgs mechanism

1) As previously the physical characteristics of particles are modelled in a representation (W, χ) of U and the associated vector bundle $W_U = U_M \times_U W$. The restriction (W, χ) to H is a representation of H. With a global section χ on X_M one has a principal fiber bundle H_M and the associated vector bundle $W_H = H_M \times_H W$ which is the restriction of W_U to H.

2) The forces fields are principal connections (**G,A**) on Q_M. **A** induces a connection $\mathbf{\Gamma}$ on X_M and a principal connection Γ_H on H_M iff $\nabla^X \varkappa = 0$.

3) The lagrangians are the same.

4) We add the variable $\chi(m) = \varphi_G(m, \varkappa(m))$ to characterize the vacuum. $\varkappa(m)$ is valued in X=U/H and with Cartan decomposition $\varkappa(m) = \exp l(m)$ where $l : M \to l_0$. The fields act on χ through the covariant derivative $\nabla^X \chi = \frac{dx}{dm} + \lambda'_g(1_U, x)\grave{A}(m)$

5) The principle of least action works in two steps :

- at the X_M level (the vacuum) : the force fields (other than gravitation) interact with the Higgs field and fix a section χ such that $\nabla^X \chi = 0$. This fixes the components a=r+1 to m of À by : $\sum_{a=r+1}^{m} \grave{A}^a \overrightarrow{\theta}_a = -R'_{\varkappa^{-1}}(\varkappa)\frac{d\varkappa}{dm}$
- at the H_M level (the system) : χ being fixed the gauge group is reduced, the fields act with the particles through Γ_H. The Lagrange equations fix the r first components of À.

This is equivalent to change the variables, and replace $\left(\grave{A}^a\right)_{a=r+1}^{m}$ by $\frac{d\varkappa}{dm}$ in the lagrangian. Thus one introduces m-r "bosons" which, besides the fermions in the Noether currents can get a mass.

6) In the standard model the Higgs mechanism is more complicated, but the scheme presented above shows the key ingredient of symmetry breakdown : a structure of the vacuum more complex than expected.

7) This mechanism has been brought up for gravitation : the SO(3,1) structure would come from a more general GL(4) structure (Sardanaschvily [23])

Part VI
APPLICATIONS

22 GENERAL RELATIVITY

1) The well known Einstein equation can be deduced by the principle of least action from a very general lagrangian. Let be :

$\mathcal{L} = \left(L_2 \left(g, z^i, z^i_\alpha \right) + a_G \left(\sum_{\alpha\beta} g^{\alpha\beta} Ric_{\alpha\beta} + \Lambda \right) \right) \sqrt{|\det g|}$

The key points are :

a) the metric g is in L_2 but not its derivatives, the other variables z^i are not involved here

b) the gravitation / gravitation interaction is modelled by the Palatini action (the cosmological constant Λ is not significant here). We have seen that in a gravitational field theory based on the metric and the Lévy-Civita connection this choice is quite mandatory.

The functional derivative with respect to $g^{\alpha\beta}$ gives :

$\frac{\delta S}{\delta g^{\alpha\beta}} = \left(\frac{\partial L_2}{\partial g^{\alpha\beta}} + a_G \frac{\partial}{\partial g^{\alpha\beta}} \left(g^{\alpha\beta} Ric_{\alpha\beta} + \Lambda \right) \right) \sqrt{|\det g|}$

$+ \left(L_2 + a_G \left(g^{\alpha\beta} R_{\alpha\beta} + \Lambda \right) \right) \frac{\partial}{\partial g^{\alpha\beta}} \sqrt{|\det g|}$

$\frac{\partial}{\partial g^{\alpha\beta}} \sqrt{|\det g|} = \frac{\partial}{\partial g^{\alpha\beta}} \left(- \det |g^{-1}| \right)^{-1/2}$

$= \frac{1}{2} \left(- \det g^{-1} \right)^{-3/2} \frac{\partial}{\partial g^{\alpha\beta}} \left(\det g^{-1} \right)$

$= \frac{1}{2} \left(- \det g^{-1} \right)^{-3/2} g_{\beta\alpha} \det g^{-1} = -\frac{1}{2} g_{\beta\alpha} \sqrt{\det |g|}$

(notice that the indexes are reversed)

$\frac{\delta S}{\delta g^{\alpha\beta}} = \left(\left(\frac{\partial L_2}{\partial g^{\alpha\beta}} + a_G Ric_{\alpha\beta} \right) - \frac{1}{2} g_{\beta\alpha} \left(L_2 + a_G \left(g^{\alpha\beta} R_{\alpha\beta} + \Lambda \right) \right) \right) \sqrt{|\det g|}$

And we get the equation :

$\left(\frac{\partial L_2}{\partial g^{\alpha\beta}} + a_G Ric_{\alpha\beta} \right) = \frac{1}{2} g_{\alpha\beta} \left(L_2 + a_G \left(g^{\lambda\mu} R_{\lambda\mu} + \Lambda \right) \right)$

$a_G \left(R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} \left(g^{\lambda\mu} R_{\lambda\mu} + \Lambda \right) \right) = -\frac{\partial L_2}{\partial g^{\alpha\beta}} + \frac{1}{2} g_{\alpha\beta} L_2$

Or with : $S_2 = \int_\Omega L_2 \sqrt{|\det g|} \varpi_0$

$\frac{\delta S_2}{\delta g^{\alpha\beta}} = \left(\frac{\partial L_2}{\partial g^{\alpha\beta}} - \frac{1}{2} g_{\alpha\beta} L_2 \right) \sqrt{|\det g|}$

we have the Einstein equation :

$$Ric_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} \left(R + \Lambda \right) = -\frac{1}{a_G \sqrt{|\det g|}} \frac{\delta S_2}{\delta g^{\alpha\beta}} \quad (115)$$

The quantity $T_{\alpha\beta} = -\frac{1}{a_G\sqrt{|\det g|}}\frac{\delta S_2}{\delta g^{ab}}$ is the "stress energy" tensor. It depends on the system and its specification is based on phenomenological assumptions about the distribution of matter and its velocity, and the other fields. It should be a symmetric 2-covariant tensor. The Einstein equation implies $\nabla^\alpha T_{\alpha\beta} = 0$. Then particles usually follow geodesics (Wald [29] p73). This equation is local and in the vacuum $T_{\alpha\beta} = 0$.

2) Our model does not use the Lévy-Civita tensor and g is not a variable, but is actually present, and the gravitational fields interaction is the same. So one can compute the stress energy tensor. However some adjustments are necessary :

a) replace g where it is actually used, meaning in the Dirac operator and the scalar product $\langle \mathcal{F}_A, \mathcal{F}_A \rangle$:

$D\psi = \sum_{\alpha ij} \left(\nabla_\alpha \psi^{ij}\right) \sum_{\beta l} g^{\alpha\beta} O'(m)^l_\beta \left(\rho \circ \Upsilon(\varepsilon_l)(e_i)\right)(m) \otimes f_j(m)$

$= \sum_{\alpha ij} \left(\nabla_\alpha \psi^{ij}\right) \sum_{\beta l} g^{\alpha\beta} O'(m)^l_\beta \left(i[\gamma_0]^p_i + \sum_{l=1}^3 [\gamma_l]^p_i\right) e_p(m) \otimes f_j(m)$

We keep : $\gamma^0 = -i\gamma_0; r = 1,2,3 : \gamma^r = \gamma_r$

$D\psi = \sum_{\alpha ij} \left(\nabla_\alpha \psi^{ij}\right) \sum_{\beta l} g^{\alpha\beta} O'^r_\beta \sum_{r=0}^3 \eta^{rr} [\gamma^r]^p_i e_p(m) \otimes f_j(m)$

$[D\psi] = \sum_{\alpha\beta r} g^{\alpha\beta} O'^r_\beta \eta^{rr} [\gamma^r] [\nabla_\alpha \psi]$

$a_F \langle \mathcal{F}_A, \mathcal{F}_A \rangle = a_F \frac{1}{2} \sum_a \sum_{\alpha\beta\lambda\mu} g^{\alpha\lambda} g^{\beta\mu} \overline{\mathcal{F}}^a_{A\alpha\beta} \mathcal{F}^a_{A\lambda\mu}$

b) In the model g is computed from O and automatically symmetric. Rather than using a constraint it is simpler to take $g_{\alpha\beta}$ and $g_{\beta\alpha}$ as distincts variables and replace $g_{\alpha\beta}$ with $\frac{1}{2}(g_{\alpha\beta} + g_{\beta\alpha})$

c) The Ricci tensor is symmetric only if g is symmetric, so one takes :

$Ric_{\alpha\beta} = \sum_{ij\gamma} [\mathcal{F}_{B\alpha\gamma}]^i_j O^\gamma_i O'^j_\beta \to \frac{1}{2} \sum_{ij\gamma} \left([\mathcal{F}_{G\alpha\gamma}]^i_j O'^j_\beta + [\mathcal{F}_{G\beta\gamma}]^i_j O'^j_\alpha \right) O^\gamma_i$

$R = \frac{1}{4} \sum_{ij\gamma} \left(g^{\alpha\beta} + g^{\beta\alpha}\right) \left([\mathcal{F}_{G\alpha\gamma}]^i_j O'^j_\beta + [\mathcal{F}_{G\beta\gamma}]^i_j O'^j_\alpha \right) O^\gamma_i$

d) With these adjusments the lagrangian becomes :

$\mathcal{L}_M = N a_M \langle \psi, \psi \rangle + N a_I \operatorname{Im} \left\langle \psi, \frac{1}{2} \sum_{\lambda\mu r} \left(g^{\lambda\mu} + g^{\mu\lambda}\right) O'^r_\mu \eta^{rr} [\gamma^r] [\nabla_\lambda \psi] \right\rangle$

$+ N a_D \sum_\lambda \langle \psi, V^\lambda \nabla_\lambda \psi \rangle \sqrt{|\det g|}$

$\mathcal{L}_F = \{ \frac{1}{8} a_F \sum_a \sum_{\lambda\mu\sigma\theta} \left(g^{\lambda\sigma} + g^{\sigma\lambda}\right) \left(g^{\mu\theta} + g^{\theta\mu}\right) \overline{\mathcal{F}}^a_{A\lambda\mu} \mathcal{F}^a_{A\sigma\theta}$

$+ \frac{1}{4} a_G \sum_{\lambda\mu ij\gamma} \left(g^{\lambda\mu} + g^{\mu\lambda}\right) \left([\mathcal{F}_{G\lambda\gamma}]^i_j O'^j_\mu + [\mathcal{F}_{G\mu\gamma}]^i_j O'^j_\lambda \right) O^\gamma_i \} \sqrt{|\det g|}$

and $L_2 = L_M + \frac{1}{8} a_F \sum_a \sum_{\lambda\mu\sigma\theta} \left(g^{\lambda\sigma} + g^{\sigma\lambda}\right) \left(g^{\mu\theta} + g^{\theta\mu}\right) \overline{\mathcal{F}}^a_{A\lambda\mu} \mathcal{F}^a_{A\sigma\theta}$

3) The functional derivatives read :

$\frac{\delta S_2}{\delta g^{\alpha\beta}} = \frac{\partial L_2 \sqrt{|\det g|}}{\partial g^{\alpha\beta}} = \left(\frac{\partial L_2}{\partial g^{\alpha\beta}} - \frac{1}{2} g_{\beta\alpha} L_2 \right) \sqrt{|\det g|}$
$\frac{\delta S_2}{\delta g^{\beta\alpha}} = \frac{\partial L_2 \sqrt{|\det g|}}{\partial g^{\beta\alpha}} = \left(\frac{\partial L_2}{\partial g^{\beta\alpha}} - \frac{1}{2} g_{\alpha\beta} L_2 \right) \sqrt{|\det g|}$
$\frac{\partial L_2}{\partial g^{\alpha\beta}} = N a_I \frac{1}{2} \sum_{\lambda\mu r} \left(\delta^\alpha_\lambda \delta^\beta_\mu + \delta^\alpha_\mu \delta^\beta_\lambda \right) O'^r_\mu \eta^{rr} \operatorname{Im} \langle \psi, [\gamma^r] [\nabla_\lambda \psi] \rangle$
$+ \frac{1}{8} a_F \sum_a \sum_{\lambda\mu\sigma\theta} \left(\left(\delta^\alpha_\lambda \delta^\beta_\sigma + \delta^\alpha_\sigma \delta^\beta_\lambda \right) \left(g^{\mu\theta} + g^{\theta\mu} \right) + \left(g^{\lambda\sigma} + g^{\sigma\lambda} \right) \left(\delta^\alpha_\mu \delta^\beta_\theta + \delta^\alpha_\theta \delta^\beta_\mu \right) \right) \overline{\mathcal{F}}^a_{A\lambda\mu} \mathcal{F}^a_{A\sigma\theta}$
$\frac{\partial L_2}{\partial g^{\alpha\beta}} = N a_I \frac{1}{2} \sum_r \eta^{rr} \left(O'^r_\beta \operatorname{Im} \langle \psi, [\gamma^r] [\nabla_\alpha \psi] \rangle + O'^r_\alpha \operatorname{Im} \langle \psi, [\gamma^r] [\nabla_\beta \psi] \rangle \right)$
$+ \frac{1}{4} a_F \sum_{a\lambda\mu} \left(\overline{\mathcal{F}}^a_{A\alpha\lambda} \mathcal{F}^a_{A\beta\mu} + \overline{\mathcal{F}}^a_{A\beta\lambda} \mathcal{F}^a_{A\alpha\mu} \right) \left(g^{\lambda\mu} + g^{\mu\lambda} \right)$
and
$\frac{\partial L_2}{\partial g^{\beta\alpha}} = N a_I \frac{1}{2} \sum_r \eta^{rr} \left(O'^r_\alpha \operatorname{Im} \langle \psi, [\gamma^r] [\nabla_\beta \psi] \rangle + O'^r_\beta \operatorname{Im} \langle \psi, [\gamma^r] [\nabla_\alpha \psi] \rangle \right)$
$+ \frac{1}{4} a_F \sum_{a\lambda\mu} \left(\overline{\mathcal{F}}^a_{A\beta\lambda} \mathcal{F}^a_{A\alpha\mu} + \overline{\mathcal{F}}^a_{A\alpha\lambda} \mathcal{F}^a_{A\beta\mu} \right) \left(g^{\lambda\mu} + g^{\mu\lambda} \right) = \frac{\partial L_2}{\partial g^{\alpha\beta}}$

4) $\frac{\delta S_2}{\delta g^{\alpha\beta}} - \frac{\delta S_2}{\delta g^{\beta\alpha}} = \frac{1}{2} \left(g_{\alpha\beta} - g_{\beta\alpha} \right) L_2 \sqrt{|\det g|}$
S_2 is symmetric with respect to $g_{\alpha\beta}$, $g_{\beta\alpha}$ so $g_{\alpha\beta} = g_{\beta\alpha}$
$\frac{1}{\sqrt{|\det g|}} \frac{\delta S_2}{\delta g^{\alpha\beta}} = N a_I \frac{1}{2} \sum_r \eta^{rr} \left(O'^r_\alpha \operatorname{Im} \langle \psi, [\gamma^r] [\nabla_\beta \psi] \rangle + O'^r_\beta \operatorname{Im} \langle \psi, [\gamma^r] [\nabla_\alpha \psi] \rangle \right)$
$+ \frac{1}{2} a_F \sum_{\lambda\mu} \left(\left(\mathcal{F}_{A\beta\lambda}, \mathcal{F}_{A\alpha\mu} \right) + \left(\mathcal{F}_{A\alpha\lambda}, \mathcal{F}_{A\beta\mu} \right) \right) g^{\lambda\mu} - \frac{1}{2} g_{\alpha\beta} \left(L_M + \frac{1}{2} a_F \sum_{\lambda\mu} \left(\mathcal{F}_{A\lambda\mu}, \mathcal{F}^{\lambda\mu}_A \right) \right)$
$T_{\alpha\beta} = \frac{1}{2} g_{\alpha\beta} \left(\frac{1}{a_G} L - R \right) - N \frac{a_I}{a_G} \frac{1}{2} \sum_r \eta^{rr} \left(O'^r_\alpha \operatorname{Im} \langle \psi, [\gamma^r] [\nabla_\beta \psi] \rangle + O'^r_\beta \operatorname{Im} \langle \psi, [\gamma^r] [\nabla_\alpha \psi] \rangle \right)$
$- \frac{1}{2} \frac{a_F}{a_G} \sum_{a\lambda\mu} \left(\left(\mathcal{F}_{A\beta\lambda}, \mathcal{F}_{A\alpha\mu} \right) + \left(\mathcal{F}_{A\alpha\lambda}, \mathcal{F}_{A\beta\mu} \right) \right) g^{\lambda\mu}$
The tensor is symmetric with respect to α, β
With $O'^r_\alpha = g_{\alpha\gamma} \eta^{rq} O^\gamma_q, O'^r_\beta = g_{\beta\gamma} \eta^{rq} O^\gamma_q$
$\sum_r \eta^{rr} \left(O'^r_\alpha \operatorname{Im} \langle \psi, [\gamma^r] [\nabla_\beta \psi] \rangle + O'^r_\beta \operatorname{Im} \langle \psi, [\gamma^r] [\nabla_\alpha \psi] \rangle \right)$
$= \sum_{rq\gamma} \left(\eta^{rr} g_{\alpha\gamma} \eta^{rq} O^\gamma_q \operatorname{Im} \langle \psi, [\gamma^r] [\nabla_\beta \psi] \rangle + \eta^{rr} g_{\beta\gamma} \eta^{rq} O^\gamma_q \operatorname{Im} \langle \psi, [\gamma^r] [\nabla_\alpha \psi] \rangle \right)$
$= \sum_{\gamma q} O^\gamma_q \left(g_{\alpha\gamma} \operatorname{Im} \langle \psi, [\gamma^q] [\nabla_\beta \psi] \rangle + g_{\beta\gamma} \operatorname{Im} \langle \psi, [\gamma^q] [\nabla_\alpha \psi] \rangle \right)$
$g^{\lambda\mu} \left(\left(\mathcal{F}_{A\beta\lambda}, \mathcal{F}_{A\alpha\mu} \right) + \left(\mathcal{F}_{A\alpha\lambda}, \mathcal{F}_{A\beta\mu} \right) \right) = 2 g^{\lambda\mu} \operatorname{Re} \left(\mathcal{F}_{A\alpha\mu}, \mathcal{F}_{A\beta\lambda} \right)$
$T_{\alpha\beta} = \frac{1}{2} g_{\alpha\beta} \left(\frac{1}{a_G} L - R \right) - \frac{a_F}{a_G} \sum_{\lambda\mu} g^{\lambda\mu} \operatorname{Re} \left(\mathcal{F}_{A\alpha\lambda}, \mathcal{F}_{A\beta\mu} \right)$
$- \frac{1}{2} \frac{a_I}{a_G} V \sum_{\gamma q} O^\gamma_q \left(g_{\alpha\gamma} \operatorname{Im} \langle \psi, [\gamma^q] [\nabla_\beta \psi] \rangle + g_{\beta\gamma} \operatorname{Im} \langle \psi, [\gamma^q] [\nabla_\alpha \psi] \rangle \right)$

5) With equation 97:
$\forall \alpha, \beta : \delta^\alpha_\beta L = N a_I \operatorname{Im} \langle \psi, \gamma^\alpha \nabla_\beta \psi \rangle + 2 a_F \sum_\lambda \operatorname{Re} \left(\mathcal{F}_{A\beta\lambda}, \mathcal{F}^{\alpha\lambda}_A \right)$
$+ 2 a_G \sum_{a\lambda} \mathcal{F}^a_{G\beta\lambda} \left(O^\alpha_{q_a} O^\lambda_{p_a} - O^\alpha_{p_a} O^\lambda_{q_a} \right)$
which is equivalent to :
$\forall q, \beta : L O'^\beta_q = N a_I \operatorname{Im} \langle \psi, \gamma^q \nabla_\beta \psi \rangle + 2 a_F \sum_{\lambda\mu} O'^q_\lambda \operatorname{Re} \left(\mathcal{F}_{A\beta\mu}, \mathcal{F}^{\lambda\mu}_A \right)$
$+ 2 a_G \sum_{a\lambda} \mathcal{F}^a_{G\beta\lambda} \left(\delta^q_{q_a} O^\lambda_{p_a} - \delta^q_{p_a} O^\lambda_{q_a} \right)$

$Na_I \operatorname{Im}\langle\psi,\gamma^q\nabla_\beta\psi\rangle = LO'^q_\beta - 2a_F\sum_{\lambda\mu}O'^q_\lambda \operatorname{Re}\left(\mathcal{F}_{A\beta\mu},\mathcal{F}_A^{\lambda\mu}\right)$
$-2a_G\sum_{a\lambda}\mathcal{F}^a_{G\beta\lambda}\left(\delta^q_{q_a}O^\lambda_{p_a}-\delta^q_{p_a}O^\lambda_{q_a}\right)$
So :
$Na_I\sum_{\gamma q}O^\gamma_q\left(g_{\alpha\gamma}\operatorname{Im}\langle\psi,[\gamma^q][\nabla_\beta\psi]\rangle + g_{\beta\gamma}\operatorname{Im}\langle\psi,[\gamma^q][\nabla_\alpha\psi]\rangle\right)$
$=\sum_{\gamma q}O^\gamma_q g_{\alpha\gamma}\{LO'^q_\beta - 2a_F\sum_{\lambda\mu}O'^q_\lambda\operatorname{Re}\left(\mathcal{F}_{A\beta\mu},\mathcal{F}_A^{\lambda\mu}\right)$
$-2a_G\sum_{a\lambda}\mathcal{F}^a_{G\beta\lambda}\left(\delta^q_{q_a}O^\lambda_{p_a}-\delta^q_{p_a}O^\lambda_{q_a}\right)\}$
$+\sum_{\gamma q}O^\gamma_q g_{\beta\gamma}\{LO'^q_\alpha - 2a_F\sum_{\lambda\mu}O'^q_\lambda\operatorname{Re}\left(\mathcal{F}_{A\alpha\mu},\mathcal{F}_A^{\lambda\mu}\right)$
$-2a_G\sum_{a\lambda}\mathcal{F}^a_{G\alpha\lambda}\left(\delta^q_{q_a}O^\lambda_{p_a}-\delta^q_{p_a}O^\lambda_{q_a}\right)\}$
$=2Lg_{\alpha\beta}-2a_F\sum_{\lambda\mu}\left(g_{\alpha\lambda}\operatorname{Re}\left(\mathcal{F}_{A\beta\mu},\mathcal{F}_A^{\lambda\mu}\right)+g_{\beta\lambda}\operatorname{Re}\left(\mathcal{F}_{A\alpha\mu},\mathcal{F}_A^{\lambda\mu}\right)\right)$
$-2a_G\sum_{a\lambda\mu}\left(g_{\alpha\mu}\mathcal{F}^a_{G\beta\lambda}+g_{\beta\mu}\mathcal{F}^a_{G\alpha\lambda}\right)\left(O^\mu_{q_a}O^\lambda_{p_a}-O^\mu_{p_a}O^\lambda_{q_a}\right)$
And the stress energy tensor reads:
$T_{\alpha\beta}=\frac{1}{2}g_{\alpha\beta}\left(\frac{1}{a_G}L-R\right)-\frac{a_F}{a_G}\sum_{\lambda\mu}g^{\lambda\mu}\operatorname{Re}\left(\mathcal{F}_{A\alpha\lambda},\mathcal{F}_{A\beta\mu}\right)$
$-\frac{1}{2}\frac{1}{a_G}\{2Lg_{\alpha\beta}-2a_F\sum_{\lambda\mu}\left(g_{\alpha\lambda}\operatorname{Re}\left(\mathcal{F}_{A\beta\mu},\mathcal{F}_A^{\lambda\mu}\right)+g_{\beta\lambda}\operatorname{Re}\left(\mathcal{F}_{A\alpha\mu},\mathcal{F}_A^{\lambda\mu}\right)\right)$
$-2a_G\sum_{a\lambda\mu}\left(g_{\alpha\mu}\mathcal{F}^a_{G\beta\lambda}+g_{\beta\mu}\mathcal{F}^a_{G\alpha\lambda}\right)\left(O^\mu_{q_a}O^\lambda_{p_a}-O^\mu_{p_a}O^\lambda_{q_a}\right)\}$
$T_{\alpha\beta}=-\frac{1}{2}g_{\alpha\beta}\frac{1}{a_G}L$
$+\frac{a_F}{a_G}\sum_{\lambda\mu}\operatorname{Re}\left(g_{\alpha\lambda}\left(\mathcal{F}_{A\beta\mu},\mathcal{F}_A^{\lambda\mu}\right)+g_{\beta\lambda}\left(\mathcal{F}_{A\alpha\mu},\mathcal{F}_A^{\lambda\mu}\right)-g^{\lambda\mu}\left(\mathcal{F}_{A\alpha\lambda},\mathcal{F}_{A\beta\mu}\right)\right)$
$+\sum_{a\lambda\mu}\left(g_{\alpha\mu}\mathcal{F}^a_{G\beta\lambda}+g_{\beta\mu}\mathcal{F}^a_{G\alpha\lambda}+\frac{1}{2}g_{\alpha\beta}\mathcal{F}^a_{G\lambda\mu}\right)\left(O^\mu_{q_a}O^\lambda_{p_a}-O^\mu_{p_a}O^\lambda_{q_a}\right)$
with $R=-\sum_{a,\alpha\beta}\mathcal{F}^a_{G\lambda\mu}\left(O^\lambda_{p_a}O^\mu_{q_a}-O^\lambda_{q_a}O^\mu_{p_a}\right)$

23 ELECTROMAGNETISM

1) The group U is here $U(1)=\{z=\exp i\theta,\theta\in\mathbb{R}\}$,an abelian group (so the bracket is null) with algebra:

$T_1U(1)=i\mathbb{R}\Rightarrow \overrightarrow{\theta}_1=i.$

The complexified $T_1U(1)^c=\mathbb{C}$ and the group $U^c=\mathbb{C}$.

The only irreducible representations are 1 complex dimensional :

$W=\left\{\sigma\overrightarrow{f},\sigma\in\mathbb{C}\right\},\chi(\exp i\theta)\overrightarrow{f}=e^{i\theta}\overrightarrow{f}$

$\chi'(1)=i=[\theta_a];[\theta_a]^t=[\theta_a]=i=-\left[\overline{\theta}_a\right]$

The state tensor is the sum of 2 right and left components : $\psi=\psi_R+\psi_L$. If each of these components is decomposable : $\psi=\phi_R\sigma_R+\phi_L\sigma_L$ where ϕ_R,ϕ_L are 2 complex dimensional vectors and σ_R,σ_L complex scalar functions.

2) The moments are :

a)

$a < 4 : P_a = \operatorname{Im}\left(\left[\phi_R^*\right]\left[\sigma_a\right]\left[\phi_L\right]\left(\sigma_L\overline{\sigma_R}\right)\right)$

$a > 3 : P_a = -\operatorname{Re}\left(\left(\left[\phi_R^*\right]\left[\sigma_{a-3}\right]\left[\phi_L\right]\right)\left(\sigma_L\overline{\sigma_R}\right)\right)$

b) $J_r = -\frac{1}{2}\left(\eta^{rr}\left(\left[\phi_R^*\right]\left[\sigma_a\right]\left[\phi_R\right]\right)\left|\sigma_R\right|^2 - \left(\left[\phi_L^*\right]\left[\sigma_a\right]\left[\phi_L\right]\right)\left|\sigma_L\right|^2\right)$

c) $\langle\psi,\psi\rangle = -2\operatorname{Im}\left(\left(\left[\phi_R\right]^*\left[\phi_L\right]\right)\left(\sigma_L\overline{\sigma_R}\right)\right)$

d) $\rho = -2\operatorname{Im}\left(\left(\left[\phi_R^*\right]\left[\phi_L\right]\right)\left(\sigma_L\overline{\sigma_R}\right)\right) = \langle\psi,\psi\rangle$

e) $\left[\mu_R - \mu_L\right]^r = \eta^{rr}\left(\left[\phi_R^*\right]\sigma_r\left[\phi_R\right]\right)\left|\sigma_R\right|^2 - \left(\left[\phi_L^*\right]\sigma_r\left[\phi_L\right]\right)\left|\sigma_L\right|^2 = -2J_r$

3) The potential À is a 1-form over M valued in the complexified, so $\grave{A} = \grave{A}_\alpha dx^\alpha, \grave{A}_\alpha \in \mathbb{C}$.

The curvature 2-form $\mathcal{F}_A = d\grave{A} = \sum_{\{\alpha\beta\}}\left(\partial_\alpha\grave{A}_\beta - \partial_\beta\grave{A}_\alpha\right)dx^\alpha \wedge dx^\beta$ and we have the first Maxwell equation : $d\mathcal{F}_A = 0$.

The equation 94 reads:

$\forall a,\alpha : -N\left(a_D V^\alpha\rho + 2ia_I\sum_r O_r^\alpha J_r\right)\det O' = 2a_F\sum_\beta\partial_\beta\left(\mathcal{F}_A^{a\alpha\beta}\left(\det O'\right)\right)$

Taking the real and imaginary parts :

$\forall a,\alpha : -a_D N\left(V^\alpha\rho\right)\det O' = 2a_F\sum_\beta\partial_\beta\left(\operatorname{Re}\mathcal{F}_A^{a\alpha\beta}\left(\det O'\right)\right)$

$\forall a,\alpha : -a_I N\sum_r O_r^\alpha J_r\det O' = a_F\sum_\beta\partial_\beta\left(\operatorname{Im}\mathcal{F}_A^{a\alpha\beta}\left(\det O'\right)\right)$

The 2nd Maxwell equation, without spin, is usually written in the General Relativity picture :

$\nabla^\beta\mathcal{F}_{A\beta\alpha} = -\mu_0 J_\alpha \Leftrightarrow \mu_0 J^\alpha\sqrt{-\det g} = \sum_\beta\partial_\beta\left(\mathcal{F}_A^{\alpha\beta}\sqrt{-\det g}\right)$

with the 4-vector current density J_α, which is for a particle $\rho\sum u^\alpha\partial_\alpha$ with the velocity measured with respect to the proper time of the particle. So we are lead to identify the real part of the field with the "usual" electromagnetic field, ρ with the electromagnetic charge and the constants are such that : $\mu_0 = \frac{a_D}{2a_F}$.

Here the electromagnetic field has a real and an imaginary part, the latter acting on the magnetic moment which is parallel to the angular momentum : $\overrightarrow{\mu} = \sum_r\left[\mu_A\right]^r\partial_r = -2\sum_r J_r\partial_r$.

Remark : it is common to have an the converse, with the "real" electromagnetic field purely imaginary. The result here is the consequence of the choice of the signature.

4) The Noether current reads here (equation 91) :

$$Y_A^\alpha = N\left(a_D\sum_\alpha V^\alpha\rho^a\partial_\alpha - ia_I\sum_r\left[\mu_R - \mu_L\right]_a^r\partial_r\right)$$
$$= N\left(a_D\sum_\alpha V^\alpha\rho\partial_\alpha - ia_I\sum_r\left[\mu_A\right]_a^r\partial_r\right)$$

because the bracket is null. So we have a global conservation of the flow of current density and magnetic moment which are its real and imaginary parts.

Furthermore equation 107 reads :$\sum_\alpha \frac{d}{d\xi^\alpha}\left(\left(N \det O'\right) V^\alpha \rho\right) = 0$ so the flow of the charge current is conserved.

5) Within the same picture the moments for symmetric states would be:

a)

$a < 4 : P_a = -\epsilon r_a \left(u^2+v^2\right) \operatorname{Im}\left(\sigma_L \overline{\sigma_R}\right)$

$a > 3 : P_a = \epsilon r_{a-3} \left(u^2+v^2\right) \operatorname{Re}\left(\sigma_L \overline{\sigma_R}\right)$

b)

$J_0 = -\frac{1}{2}\left(u^2+v^2\right)\left(\left|\sigma_R\right|^2 + \left|\sigma_L\right|^2\right); r > 0 : J_r = \frac{1}{2}\left(u^2+v^2\right)\epsilon r_r \left(\left|\sigma_R\right|^2 - \left|\sigma_L\right|^2\right)$

$J_r = -\frac{1}{2}\left(u^2+v^2\right)\epsilon r_r \left(\eta^{rr}\left|\sigma_R\right|^2 - \left|\sigma_L\right|^2\right)$ with $r_0 = \epsilon$

c) $\langle\psi,\psi\rangle = -2\left(u^2+v^2\right) \operatorname{Im}\left(\sigma_L \overline{\sigma_R}\right)$

d) $\rho = 2\left(u^2+v^2\right) \operatorname{Re}\left(\left[\sigma_L\right]\left[i\right]^t\left[\sigma_R\right]^*\right) = -2\left(u^2+v^2\right) \operatorname{Im}\left(\sigma_L \overline{\sigma_R}\right)$

e) $\left[\mu_R - \mu_L\right]^r = -2J_r$

So the particle has a charge if $\operatorname{Im}\left(\sigma_L \overline{\sigma_R}\right) \neq 0$, this emphasizes the need to use of the complexified of U(1) (with U(1) the charge would be null) and of different functions for the right and the left side (we know that the electromagnetic field is part of the larger electroweak field for which chirality is crucial). The quantities (r_1, r_2, r_3) give the spatial direction of both the angular and the magnetic momentum. It is also the orientation of the would be linear momentum, up to a sign. Equation 109 reads :

$a_D \operatorname{Im}\left\langle \psi, \frac{d\psi}{d\xi^0}\right\rangle = 2 \operatorname{Im}\left(\sigma_L \overline{\sigma_R}\right)\left(a_M + a_D \sum_{\alpha a} V^\alpha \operatorname{Re} \grave{A}_\alpha\right)$

$+a_D \sum_a V^\alpha \epsilon \sum_{a=1}^3 \left(r_a \left(\operatorname{Im}\left(\sigma_L \overline{\sigma_R}\right) G_\alpha^a - \operatorname{Re}\left(\sigma_L \overline{\sigma_R}\right) G_\alpha^{a+3}\right)\right)$

$+a_I \frac{1}{2}\epsilon\left(\eta^{jj}\left|\sigma_R\right|^2 - \left|\sigma_L\right|^2\right)\left(\sum_{\alpha j}\left(\frac{dN(\det O')O_j^\alpha}{N(\det O')d\xi^\alpha} + 2\left(\operatorname{Im} \grave{A}_j\right)\right) r_j + \left(\left[r\right]\left[G_r\right]\right)_r\right)$

with $\left(u^2+v^2\right) = 1$

CONCLUSION

Let us sum up the main results :

1) It is possible to model a system with individually interacting particles, with gravitation and other fields,using the modern concepts of physical theory (Yan-Mills connections, fiber bundle, Clifford algebra), but standing in the classical picture. The principle of least action can be implemented, and the constraints on the lagrangian can be met.

2) It is possible to give a sensible description of gravitation in the general connection and tetrad framework, without involving a metric. This opens the path to more general solutions than the Lévy-Civita connection, that would be required if, as it seems, the connection is not torsion free. Moreover the calculations are manageable, and can give explicit solutions with respect to natural variables (the structure coefficients).

3) In the simple model we have seen the crucial role of "moments", clearly identified with respect to the state tensor and clearly related to basic physical concepts. Noether currents supply the conservation equations useful to a further study.

4) The framework used to describe particles and fields provides a good basis to study symmetries, and give hints for a better understanding of some "paradoxical quantum phenomenon".

The main outcome of this paper is probably pedagogical, as it covers a great deal of concepts in theoretical physics, using the tools of the trade. But beyond this, several issues would be worth of further studies.

1) Is it possible to implement the machinery in the pure gravitational case ? So far General Relativity has suffered both from intractable calculations, and the metric obsession. It would be immensely useful to have manageable models, pertinent for the hottest topics such as the movements of large systems (galaxies notably) which are, after all, at the core of the "dark matter issue".

2) That is good to be able to model the individual movements of particles, but of course it is mostly theoretical. So the next step is to introduce some probabilistic particles distributions, this should be easy using the initial state represented by the f function. In the thermodynamic picture it would be of great interest to find a link between the moments and the "function of state" of the whole system.

3) If this construction makes any sense, does it provide us with a better understanding of the fundations of quantum mechanics ? I think so, and it will be the topic of a next paper.

Some last words on more technical issues :

1) Introduce the velocity in the lagrangian is possible, even in the General Relativty picture, and probably mandatory. It clearly enhances the understanding of the interactions, showing the crucial and distinctive role of the kinematic and dynamic parts. From this point of view the Dirac operator, as essential as it is, is not enough as it emphasizes the first part. The solution implemented here can certainly be improved.

2) Complex fields are mandatory, and they deserve the full treatment, even if it is cumbersome. Any shortcut is hazardous.

3) The issue of the signature of the metric is still open...

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